

PATH-DEPENDENT OPTION PRICING WITH EXPLICIT SOLUTIONS, STOCHASTIC APPROXIMATION AND HESTON EXAMPLES

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ABSTRACT. New simulation approaches to evaluating path-dependent options without matrix inversion issues nor Euler bias are evaluated. They employ three main contributions: Stochastic approximation replaces regression in the LSM algorithm; Explicit weak solutions to stochastic differential equations are developed and applied to Heston model simulation; and Importance sampling expands these explicit solutions. The approach complements Heston (1993) and Broadie and Kaya (2006) by handling the case of path-dependence in the option's execution strategy. Numeric comparison against standard Monte Carlo methods demonstrate up to two orders of magnitude speed improvement. The general ideas will extend beyond the important Heston setting.

1. INTRODUCTION

The optimal pricing of American and other path-dependent options for multiple factor models remains problematic. Traditionally, finite difference methods have been used (see e.g. Schwartz (1977), Wilmott et. al. (1995)) to solve the corresponding partial differential equation. However, they can be computationally expensive when the model has multiple factors and also complicated to adapt when the model has jumps. This has led to the development and use of Monte Carlo based pricing methods (see e.g. Boyle (1977), Duffie and Glynn (1995), Boyle et. al. (1997), Carriere (1996)), for which one needs simulation. A most successful simulation method for Monte Carlo multi-factor, path-dependent option pricing is the LSM algorithm developed by Longstaff and Schwartz (2001) and further analyzed by Clément et. al. (2002). As usual, they approximate American (and other continuously-executable) options discretely, implementing and analyzing the resulting Bermuda-style options. However, there are problems.

1.1. Motivational Problem. Suppose we wanted to price an American (really Bermudan) Put option based upon the Heston model (see (1.5) to follow) with Heston and option parameters: $\nu = \frac{8.1\kappa^2}{4}$, $\mu = 0.0319$, $\rho = -0.7$, $\varrho = 6.21$, $\kappa = 0.2$, option duration $T = 50$, initial price $S_0 = 100$, initial volatility $V_0 = 0.102$, and the

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strike price $K = 100$. The real ground truth price of this option will turn out to be \$7.9426. However, if we use the LSM algorithm with Euler (or Implicit Milstein as proposed by Kahl and Jäckel (2006)) Monte Carlo simulation, then the best we can get on an inexpensive contemporary computer is \$7.371 for as we try to go beyond that the algorithm fails numerically, producing smaller values while taking longer times to compute. Our goals herein are to get around the numeric least squares regression problems of the LSM algorithm and the slow, biased nature of the Euler and Milstein¹ simulation methods. We do this by explicit weak solutions and stochastic approximation. The result will be a *three order of magnitude speed improvement* in simulation and a *two order of magnitude speed improvement* in path-dependent option pricing.

1.2. The LSM/Simulation Setting. Suppose there is a complete filtered (risk-neutral) probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^T, P)$ ² supporting a Markov chain $\{(S_t, V_t)\}_{t=0}^T$ with state space $D = D_S \times D_V$, representing the observable and hidden components of the asset state (like price and volatility), as well as the (discounted) adapted payoff $Z_t \geq 0$ received for executing the option at time $t \in [0, T]$. Then, the option-pricing objective is to compute $\sup_{\tau_0 \in \mathcal{T}_{0,T}} E[Z_{\tau_0}]$, where $\mathcal{T}_{t,T}$ denotes the collection of stopping times with values in $\{t, t+1, \dots, T\}$. Using dynamic programming, one finds (see Clément et. al. (2002)) a best $\tau_0 \in \mathcal{T}_{0,T}$ by working backwards according to

$$\begin{cases} \tau_T &= T \\ \tau_t &= t 1_{\{Z_t \geq E[Z_{\tau_{t+1}} | \mathcal{F}_t]\} \cap \{Z_t > 0\}} + \tau_{t+1} 1_{\{Z_t < E[Z_{\tau_{t+1}} | \mathcal{F}_t]\} \cup \{Z_t = 0\}} \quad \forall t < T \end{cases}.$$

Typically, $E[Z_{\tau_{t+1}} | \mathcal{F}_t] > 0$ so $\cap \{Z_t > 0\}$ and $\cup \{Z_t = 0\}$ do not effect the recursion.

Now, assume:

Total: there are measurable real-valued functions $(f_t)_{t=0}^T$ and $(e_k)_{k=1}^\infty$ on D such that $E[Z_{\tau_t} | \mathcal{F}_t] = f_t(S_t, V_t)$ for all $t = 0, \dots, T$ and $\{e_k(S_t, V_t)\}_{k=1}^\infty$ is total³ on $L^2(\sigma(S_t, V_t), 1_{\{Z_t > 0\}} dP)$ for all $t = 1, \dots, T-1$.

Following Longstaff and Schwartz (2001) to create the $(e_k)_{k=1}^\infty$, we often start with bases functions $(e_k^S)_{k=1}^\infty, (e_k^V)_{k=1}^\infty$ on $L^2(D_S), L^2(D_V)$ respectively and let $(e_k(s, v))_{k=1}^\infty$ be some ordering of $\{e_{k_1}^S(s) e_{k_2}^V(v)\}_{k_1, k_2=1}^\infty$.

The key idea in the LSM algorithm is to estimate the conditional expectations $E[Z_{\tau_t} | \mathcal{F}_t]$ (by first estimating $E[Z_{\tau_{t+1}} | \mathcal{F}_t]$) from the cross-sectional data using projection P_t^J onto the closed linear span of $\{e_k(S_t, V_t)\}_{k=1}^J$ and least-squares regression. Indeed, (Clément et. al. , 2002, Theorem 3.1) show that

$$\lim_{J \rightarrow \infty} E[Z_{\tau_t}^J | \mathcal{F}_t] = E[Z_{\tau_t} | \mathcal{F}_t] \quad (1.1)$$

¹Throughout this paper references to the Milstein method will always mean the Implicit Milstein method proposed by Kahl and Jäckel as the normal Milstein method does not perform well.

²In many settings there are multiple risk-neutral measures and one is chosen by calibrating to model market data. In the Heston case, the volatility component causes the non-uniqueness and should be calibrated using e.g. real option prices. We assume throughout that this has been done.

³A subset of a Hilbert space is total if its span is the entire space.

in L^2 for all $t \in \{0, \dots, T\}$, where

$$\begin{cases} \tau_T^J &= T \\ \tau_t^J &= t 1_{\{Z_t \geq P_t^J[Z_{\tau_{t+1}^J}]\} \cap \{Z_t > 0\}} + \tau_{t+1}^J 1_{\{Z_t < P_t^J[Z_{\tau_{t+1}^J}]\} \cup \{Z_t = 0\}} \quad \forall t < T \end{cases} .$$

Then, letting $e^J = (e_1, \dots, e_J)'$ (where a' denotes transpose of vector or matrix a) and assuming

Non-singular: $E[e^J(S_t, V_t)(e^J(S_t, V_t))' 1_{\{Z_t > 0\}}]$ is positive definite,

Longstaff and Schwartz (2001) recognize that the α_t^J in $P_t^J[Z_{\tau_{t+1}^J}] = \alpha_t^J \cdot e^J(S_t, V_t)$ is $\alpha_t^J = E[e^J(S_t, V_t)(e^J(S_t, V_t))' 1_{\{Z_t > 0\}}]^{-1} E[Z_{\tau_{t+1}^J} e^J(S_t, V_t) 1_{\{Z_t > 0\}}]$ i.e. the solution to

$$\min_{\alpha^J} E[|Z_{\tau_{t+1}^J} - \alpha^J \cdot e^J(S_t, V_t)|^2 1_{\{Z_t > 0\}}], \quad (1.2)$$

which they solve by Monte Carlo simulation and linear regression: Let $\{(S^j, V^j, Z^j)\}_{j=1}^N$ be i.i.d. copies of (S, V, Z) and the $\tau_{t+1}^{J,j}$ satisfy

$$\begin{cases} \tau_T^{J,j} &= T \\ \tau_t^{J,j} &= t 1_{\{Z_t^j \geq P_t^J[Z_{\tau_{t+1}^{J,j}}^j]\} \cap \{Z_t^j > 0\}} + \tau_{t+1}^{J,j} 1_{\{Z_t^j < P_t^J[Z_{\tau_{t+1}^{J,j}}^j]\} \cup \{Z_t^j = 0\}} \quad \forall t < T \end{cases} . \quad (1.3)$$

Then, their least squares estimate is $\alpha_t^{J,N} = (A_t^N)^{-1} b_t^N$ with

$$A_t^N = \frac{1}{N} \sum_{j=1}^N e^J(S_t^j, V_t^j) e^J(S_t^j, V_t^j)' 1_{Z_t^j > 0}, \quad b_t^N = \frac{1}{N} \sum_{j=1}^N Z_{\tau_{t+1}^{J,j}}^j e^J(S_t^j, V_t^j) 1_{Z_t^j > 0}.$$

Notice that $\tau_t^{J,j}$ depends on $P_t^J[Z_{\tau_{t+1}^{J,j}}^j]$ which depends upon $\alpha_t^{J,N}$ which in turn depends upon $\tau_{t+1}^{J,j}$, meaning we must construct these objects in reverse time and at each time compute $\alpha_t^{J,N}$ prior to $\tau_t^{J,j}$.

1.3. Weaknesses of Current Methods. The LSM algorithm has a weakness: The regression requires inverting a (generally) dense $J \times J$ matrix A_t^N with random coefficients, which becomes ill-conditioned as the number of factors in the model or the desired accuracy (and consequently the number of bases functions J required) increases. Many examples given in Longstaff and Schwartz (2001) have features that may allow a lower number of basis functions⁴: Shorter durations facilitate a smaller J because there are fewer possible execution times to choose from in the Bermudian approximations. Single factor models make projection one dimensional, which generally facilitates better approximation with fewer functions versus higher dimensional projection. American put options with strike price K effectively restrict S to $[0, K]$ or less, which also makes the projection “easier”. The need for lower accuracy reduces the required J as it becomes acceptable to get more of the optimal stopping possibilities wrong. Not all problems have these features. In some examples below, J will be large enough that matrix inversion is problematic. Fortunately,

⁴The most used in Longstaff and Schwartz (2001) was 26. We will often use many more in the examples below.

there is a stochastic approximation alternative and it is also faster than regression. This is the first main contribution of this paper.

The other major problems with the simulation approach to path-dependent option pricing are computation time and bias. The famous geometric Brownian motion (GBM) model, utilized in the classical Black-Scholes option pricing formula (see Black and Scholes (1973), Merton (1973)), has constant volatility and follows the linear Itô stochastic differential equation (SDE)

$$dS_t = \mu S_t dt + \kappa S_t dB_t, \quad (1.4)$$

where B is a standard Brownian motion and μ , κ are the drift and volatility parameters. It is well known that the GBM model is overly simplistic, results in unnatural phenomena like the volatility smile commonly observed in market option prices (see Jackwerth and Rubinstein (1996) for a detailed survey) and should be replaced by stochastic volatility (SV) models with two components: price S and stochastic variance V (or volatility $V^{\frac{1}{2}}$) that replaces the constant κ in the GBM model.

Heston (1993) introduced a stochastic volatility model with closed form European-call-option prices for stock, bond and foreign currency spot prices. Let B , β to be (scalar) independent standard Brownian motions. Then, the *Heston* model is:

$$d \begin{pmatrix} S_t \\ V_t \end{pmatrix} = \begin{pmatrix} \mu S_t \\ \nu - \varrho V_t \end{pmatrix} dt + \begin{pmatrix} \sqrt{1 - \rho^2} S_t V_t^{\frac{1}{2}} & \rho S_t V_t^{\frac{1}{2}} \\ 0 & \kappa V_t^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} dB_t \\ d\beta_t \end{pmatrix}, \quad (1.5)$$

with parameters $\mu \in \mathbb{R}$, $\rho \in [-1, 1]$ and $\nu, \varrho, \kappa > 0$. The volatility component is just the Cox-Ingersoll-Ross (CIR) model. The volatility can hit 0 when $\nu < \frac{\kappa^2}{2}$ and can still approach 0 when the Feller condition $\nu \geq \frac{\kappa^2}{2}$ holds. From a financial perspective, hitting zero would imply randomness coming out of the price, which is not common, so we generally have ν larger than $\frac{\kappa^2}{2}$. An important feature of the Heston model is that it allows arbitrary correlation $\rho \in [-1, 1]$ between volatility and spot asset returns. Indeed, ρ is often negative in financial markets (see e.g. Fouque et. al. (2000, p. 41) or Yu (2005)). The Heston model can be used to explain and correct for skewness and strike price bias and to outperform other popular SV models on real data (see Kouritzin (2015) for the later). Broadie and Kaya (2006) developed an exact (without bias) simulation method for the Heston model to price options with at most weak path dependence. This paper addresses the remaining significant difficulty, effectively pricing path-dependent Heston options including the American and Asian options. Herein, the Heston model stochastic differential equations (sdes) are solved explicitly in weak form and these solutions are used to price options and do Monte Carlo simulations.

The Euler-Maruyama and Milstein simulation methods have obvious problems for the Heston model: 1) While the process itself is nonnegative, the discretization may try producing negative values causing evaluation issues when square rooted. 2) The rate of convergence to the actual diffusion is slow. In fact, Broadie and Kaya (2006) did a nice job of demonstrating the bias problem of these methods even when the computations are appropriately balanced in the sense of Duffie and Glynn (1995).

3) The computation time is large, making real-time application more difficult for higher-volume, rapidly-traded equities. For example, the use of Euler-Maruyama and Milstein methods made real-time application (versus back data study) impossible in Kouritzin (2015). Hence, exact simulation (c.f. Broadie and Kaya (2006)), where Heston model specifics are used to avoid bias and increase speed, is desired. Unfortunately, this type of exactness (in terms of distribution transforms) is not amenable to valuing American, Asian and other heavily-path-dependent options. Herein, we introduce explicit weak solutions to the Heston SDEs, our most significant contribution, which makes simulation and Monte Carlo path-dependent option pricing relatively easy. For expository reasons, we keep our goals balanced. We introduce new pricing algorithms, give new theorems for explicit solutions, develop new methods for finding explicit solutions and provide American and Asian option pricing examples. We could have gone further in these directions but that might have detracted from our new ideas. A proper convergence rate analysis of our algorithms is left to future work.

1.4. Layout. The remainder of this paper is laid out as follows: Our new algorithms and theoretical results are given in Section 2. The first algorithm is a stochastic approximation variation of the LSM algorithm. The second algorithm is for simulating Heston SDEs. It fits into the first algorithm when the Heston model is used and is based upon our main theorems. The first theorem gives basic explicit solutions that hold under a restriction on the parameters of the Heston model. The second result provides weak solutions when this restriction does not hold. Section 3 compares our new Heston simulation algorithms to the Euler-Maruyama and Milstein simulation methods and shows a *three order* of magnitude speed improvement for the same accuracy. Section 4 compares our new Heston simulation and SA algorithms to the LSM algorithm as well as the Euler-Maruyama and Milstein simulation methods on the American and Asian option pricing problems. In particular, pricing of put, call and straddle options are considered for the Heston model and the combined effect of the new simulation and SA algorithms are shown to provide a *two order* of magnitude improvement on pricing such options compared to the standard LSM/Euler or LSM/Milstein approach. Our conclusions are in Section 5 and our proofs are relegated to the appendix, which is Section 6. However, these proofs are really our method of finding explicit (weak) solutions for financial models. Hence, they could turn out to be the most important part of this work.

2. ALGORITHMS AND RESULTS

2.1. Stochastic Approximation Pricing Algorithm. Stochastic Approximation (SA) algorithms solve stochastic optimization problems like the mean-square optimization problem (1.2). The first, and most famous, SA algorithms are the Robbins-Monro and Kiefer-Wolfowitz algorithms introduced respectively in Robbins and Monro (1951) and Kiefer and Wolfowitz (1952). Our application is similar to the SA framework of Kouritzin (1996) and Kouritzin and Sadeghi (2015). Suppose $\{(L^j, S^j, V^j, Z^j)\}_{j=1}^N$ are i.i.d. copies of (L, S, V, Z) , where S, V, Z are as in the introduction and L is

some likelihood, i.e. a non-negative martingale and satisfying $E[L_t] = 1$ for all t . L 's purpose is to reweight (S, V, Z) so they have the correct joint process distribution with respect to a new probability measure \hat{P} when they do not under P . This facilitates efficient simulation as will become clear in the sequel. (The reader can take $L^j = L = 1$ on first reading so we are back to the situation considered in Longstaff and Schwartz (2001).) Now, we generalize A_t^N and b_t^N to

$$A_t^N = \frac{1}{N} \sum_{j=1}^N A_j, \text{ where } A_j = \frac{L_t^j e^J(S_t^j, V_t^j) e^J(S_t^j, V_t^j)' 1_{Z_t^j > 0}}{\frac{1}{N} \sum_{i=1}^N 1_{Z_t^i > 0}},$$

$$b_t^N = \frac{1}{N} \sum_{j=1}^N b_j, \text{ where } b_j = \frac{L_t^j Z_{\tau_{t+1}^j}^j e^J(S_t^j, V_t^j) 1_{Z_t^j > 0}}{\frac{1}{N} \sum_{i=1}^N 1_{Z_t^i > 0}}.$$

Then, it follows from the (exchangeable) strong law of large numbers⁵ that

$$\text{SLLN-A: } \lim_{N \rightarrow \infty} A_t^N = \frac{E[L_t e^J(S_t, V_t) e^J(S_t, V_t)' 1_{Z_t > 0}]}{P(Z_t > 0)} = \frac{\hat{E}[e^J(S_t, V_t) e^J(S_t, V_t)' 1_{Z_t > 0}]}{P(Z_t > 0)}$$

$$\text{SLLN-b: } \lim_{N \rightarrow \infty} b_t^N = \frac{E[L_t Z_{\tau_{t+1}}^J e^J(S_t, V_t) 1_{Z_t > 0}]}{P(Z_t > 0)} = \frac{\hat{E}[Z_{\tau_{t+1}}^J e^J(S_t, V_t) 1_{Z_t > 0}]}{P(Z_t > 0)},$$

where $\left. \frac{d\hat{P}}{dP} \right|_{\mathcal{F}_t} = L_t$ and \hat{E} denotes expectation with respect to new probability measure \hat{P} . Under similar conditions Kouritzin (1996)⁶ establishes that $\lim_{N \rightarrow \infty} \alpha_t^{J,N} = \alpha_t^J$ a.s. $[\hat{P}]$ (and therefore a.s. $[P]$) for any $\gamma > 0$, where $\alpha_t^{J,j}$ is defined recursively by: $\alpha_t^{J,0} = 0$ and $k = 1$ initially and then for $j = 1, 2, \dots, N$:

$$(\alpha_t^{J,j}, k) = \begin{cases} (\alpha_t^{J,j-1}, k) & Z_t^j = 0 \\ (\alpha_t^{J,j-1} + \frac{\gamma L_t^j}{k} (Z_{\tau_{t+1}^j}^j - e^J(S_t^j, V_t^j)' \alpha_t^{J,j-1}) e^J(S_t^j, V_t^j), k+1) & Z_t^j > 0 \end{cases}.$$

Recall here that (S, V, Z) has the desired distribution under \hat{P} not P so

$$\alpha_t^J = \hat{E}[e^J(S_t, V_t) e^J(S_t, V_t)' 1_{Z_t > 0}]^{-1} \hat{E}[Z_{\tau_{t+1}}^J e^J(S_t, V_t) 1_{Z_t > 0}].$$

Hence, we obtain convergence to the same solution as the least-squares regression method but without numerically nasty matrix inversion. Substituting $\lim_{N \rightarrow \infty} \alpha_t^{J,N} = \alpha_t^J$ a.s. into the work of Clément et. al. (2002) yields (after a small amount of work) convergence in probability (at least) for this option pricing procedure. Moreover, Kouritzin and Sadeghi (2015) and Kouritzin (1994) could be used to obtain a.s.

⁵The standard i.i.d. strong law does not apply since $\tau_{t+1}^{J,j}$ depends weakly on the other particles through the projection estimate. Still, this dependence dies out fast enough as $N \rightarrow \infty$ that a general strong law does apply.

⁶The triangle nature (through dependence of $\alpha_t^{J,j}$ on the number of particles N) of the summed terms in b_t^N was not considered in this work. However, the proof will still work in this case.

rates of convergence and rates of r^{th} -mean convergence respectively for our parameter estimates if Conditions SLLN-A and SLLN-b are replaced by slightly stronger conditions (that would still hold in our setting).

Our first contribution is a numerically stable alternative to LSM algorithm of Longstaff and Schwartz (2001). In particular, the following SA algorithm will be a big improvement when J is not very small.

Initialize: Fix functions e_k and $\gamma > 0$; set $\zeta = \lambda = 0$, all $\alpha_t^J = 0$ and all $\tau^{J,j} = T$.

Simulate: Create independent copies $\{L^j, S^j, V^j, Z^j\}_{j=1}^N$ of (L, S, V, Z) .

Repeat: for $t = T - 1$ down to 0:

$k = 0$

Repeat: for $j = 1$ to N :

Stochastic Approximation: If $Z_t^j > 0$ then $k = k + 1$ and

$$\alpha_t^J = \alpha_t^J + \frac{\gamma L_t^j}{k} (Z_{\tau^{J,j}}^j - e^J(S_t^j, V_t^j)' \alpha_t^J) e^J(S_t^j, V_t^j)$$

Repeat: for $j = 1$ to N :

Adjust Stopping Times: If $Z_t^j > 0$ and $Z_t^j \geq \alpha_t^J \cdot e^J(S_t^j, V_t^j)$, then $\tau^{J,j} = t$

Price Option:

Repeat: for $j = 1$ to N :

$$\zeta = \zeta + L_{\tau^{J,j}}^j Z_{\tau^{J,j}}^j$$

$$\lambda = \lambda + L_{\tau^{J,j}}^j$$

Value: $O = \frac{\zeta}{\lambda}$

Remark 1. For each $\{(L_t^j, S_t^j, V_t^j, Z_t^j), t = 0, 1, \dots, T\}$, L^j is a non-negative martingale with mean 1, $\{(S_t^j, V_t^j), t = 0, 1, \dots, T\}$ has the desired risk-neutral (process) distribution and $\{Z_t^j, t = 0, 1, \dots, T\}$ is the discounted payoff process with respect to probability $\hat{P}^j(A) = E[L_T^j 1_A]$. The preferred method to create these simulations for the Heston and other models with explicit weak solutions follows in Subsection 2.3. In this case, $L_t^j = \hat{L}_{t \wedge \eta_\varepsilon}^j$ where \hat{L}^j and η_ε are defined in Subsection 2.3.

Remark 2. This procedure is set up to be convenient for American options. However, it is easy to adjust it to Asian options. If this is desired, then we would simulate the running average price R_t^j as well (see Remark 10 to follow). These average prices would become the S^j 's in this procedure, while the spot price would become part of the V^j 's. For example, in our Heston case each V^j would be the whole 2-dimensional model and the new S^j would just be the average price as explained in Remark 10.

Remark 3. The SA algorithm gain $\gamma > 0$ can effect performance due to the finiteness of our particle system. The better choices are dependent upon the model parameters and will be given on an example-by-example basis. In fact, a more general step size $\frac{\gamma}{k^\alpha}$ in place of $\frac{\gamma}{k}$ (see Kouritzin and Sadeghi (2015) for a discussion), a (positive definite) matrix-valued γ or a two step algorithm like that introduced

in Polyak and Juditsky (1992) is often desired. None of these improvements are considered in this introductory paper. We just choose a reasonable scalar γ .

Remark 4. The first J Haar bases functions on $[0, K]$ can be a good choice of $(e_k^S)_{k=1}^J$ for a price only model and a put option with strike price K . For volatility in Heston-type models, we can adapt the Haar bases to $[0, \infty]$. Specifically, letting h_k be the k^{th} Haar function on $[0, 1]$, we can rescale by letting $e_k^V(x) = \sqrt{s'(x)}h_k(s(x))$ for some differentiable scale function s satisfying $s(0) = 0$ and $\lim_{x \rightarrow \infty} s(x) = 1$ to obtain new bases functions $\{e_k^V\}_{k=1}^J$ on $[0, \infty]$. An example is $s(x) = \frac{x}{1+x}$ so $e_k^V(x) = \frac{1}{1+x}h_k\left(\frac{x}{1+x}\right)$. Naturally, there are other good scalings and choices of (e_k^V) . Indeed, we will use the weighted Laguerre functions below since that is what Longstaff and Schwartz (2001) used.

Remark 5. We call this algorithm the SA or SA pricing algorithm. Our version of the LSM algorithm is obtained simply by replacing the Stochastic Approximation part by the following Least Squares Regression:

$k = 0$

Repeat: for $j = 1$ to N :

Least Squares Regression: If $Z_t^j > 0$ then $k = k + 1$ and

$$\begin{aligned} A_t^J &= \frac{k-1}{k}A_t^J + \frac{L_t^j}{k}e^J(S_t^j, V_t^j)e^J(S_t^j, V_t^j)' \\ b_t^J &= \frac{k-1}{k}b_t^J + \frac{L_t^j}{k}Z_{\tau^j, j}^j e^J(S_t^j, V_t^j) \end{aligned}$$

$$\alpha_t^J = (A_t^J)^{-1}b_t^J.$$

We also set all $A_t^J = 0$ (matrix of all zeros) and $b_t^J = 0$ during the initialization. The rest of the algorithm is the same.

2.2. Explicit and Weighted Solutions. There are several papers on exact simulation for the Heston model (see e.g. Andersen (2007), van Haastrecht and Pelsser (2010)). Most of these contributions build off of Broadie and Kaya (2006) and/or rely on a change of variables as well as Feller's characterization of the transition function for the square root diffusion. Generic difficulties of these methods are:

- Algorithm complexity - often involving numeric convergence.
- Accommodating all possibly desired drifts.
- Allowing derivative payoffs that depend on the underlying asset at many points in time.
- Admitting time dependence in the spot price variance.
- Handling the volatility approaching or hitting 0.

Alternatively, one should consider the possibility of *explicit* representations of the Heston sdes as a time-dependent function $\phi\left(\int_s^t U_u dW_u, t\right)$ of a simple Gaussian stochastic integral. It is discovered in our companion paper Kouritzin and Remillard

(2016)⁷ that a necessary and sufficient condition for the Itô SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad (2.1)$$

to have a strong solution with such an explicit representation locally (for some drift coefficient b) is the diffusion coefficient columns σ_j satisfy the Lie bracket condition:

$$(\nabla \sigma_i)\sigma_j = (\nabla \sigma_j)\sigma_i \quad \forall i, j. \quad (2.2)$$

Unfortunately, the Heston model does not satisfy (2.2) since

$$(\nabla \sigma_1)\sigma_2 = \begin{pmatrix} sv\rho\sqrt{1-\rho^2} + \frac{s\kappa\sqrt{1-\rho^2}}{2} \\ 0 \end{pmatrix} \neq \begin{pmatrix} sv\rho\sqrt{1-\rho^2} \\ 0 \end{pmatrix} = (\nabla \sigma_2)\sigma_1$$

when $\sigma = (\sigma_1 \sigma_2) = \begin{pmatrix} \sqrt{1-\rho^2}sv^{\frac{1}{2}} & \rho sv^{\frac{1}{2}} \\ 0 & \kappa v^{\frac{1}{2}} \end{pmatrix}$, where s and v represent the state variables for price and variance (square of volatility). Hence, we will have to consider weak solutions to get an explicit representation for the Heston sdes. While our focus herein is largely on solving the sdes and using the solutions in simulation for option pricing, the solutions can also be used in other ways.

Explicit solutions are fragile. For example, it is shown in Kouritzin (2000) that scalar SDEs only have explicit solutions for specific drift coefficients. Hence, it is reasonable to expect a condition on the Heston model parameters for an explicit solution (if they are even possible). This condition is:

$$\mathbf{C}: \nu = \frac{n\kappa^2}{4} \text{ for some } n = 1, 2, 3, \dots$$

Fortunately, this is all that is needed.

Theorem 1. *Suppose $n \in \{1, 2, 3, 4, \dots\}$, Condition (C) holds with this n and W^1, \dots, W^n, B are independent standard Brownian motions. Then, the Heston (price and volatility) model (1.5) has explicit weak solution:*

$$S_t = S_0 \exp \left(\sqrt{1-\rho^2} \int_0^t V_s^{\frac{1}{2}} dB_s + \left[\mu - \frac{\nu\rho}{\kappa} \right] t + \left[\frac{\rho\rho}{\kappa} - \frac{1}{2} \right] \int_0^t V_s ds + \frac{\rho}{\kappa} (V_t - V_0) \right), \quad (2.3)$$

$$V_t = \sum_{i=1}^n (Y_t^i)^2, \quad (2.4)$$

where $\{Y_t^i = \frac{\kappa}{2} \int_0^t e^{-\frac{\kappa}{2}(t-u)} dW_u^i + e^{-\frac{\kappa}{2}t} Y_0^i\}_{i=1}^n$ are Ornstein-Uhlenbeck processes and

$$\beta_t = \sum_{i=1}^n \int_0^t \frac{Y_u^i}{\sqrt{\sum_{j=1}^n (Y_u^j)^2}} dW_u^i$$

is the other Brownian motion appearing in (1.5).

⁷Earlier methods of explicitly expressing sde solutions in terms of the driving Brownian motion include Doss (1977), Sussmann (1978), Yamato (1979), Kunita (1984), Kouritzin and Li (2000) and Kouritzin (2000).

While the drift and diffusion coefficients do not satisfy the classical conditions for a strong solution, it follows from Remark 1.1 of Bass and Perkins (2002) as well as Rogers and Williams (1987) that it does have a weak solution. Theorem 1 also establishes weak solutions but, most importantly, also gives them explicitly in a computable way.

Proof. See Appendix. \square

Remark 6. *The solution is valid for any $\{Y_0^i\}_{i=1}^n$ such that $\sum_{i=1}^n (Y_0^i)^2 = V_0$. By expanding the squares, V_t can be written as $V_t = V_t^X + V_t^G + V_t^D$, the sum of a χ^2 random variable plus a Gaussian variable plus a deterministic piece. In particular, the moment generating functions of the first two pieces are:*

$$M_{V_t^X}(\theta) = \left(1 - \frac{\kappa^2}{2\varrho} [1 - e^{-\varrho t}] \theta\right)^{-\frac{n}{2}} \text{ and } M_{V_t^G}(\theta) = \exp\left(\frac{V_0 \kappa^2}{2\varrho e^{\varrho t}} [1 - e^{-\varrho t}] \theta^2\right) \quad (2.5)$$

(for θ in a neighbourhood of 0) while the deterministic piece is just

$$V_t^D = \exp(-\varrho t) V_0. \quad (2.6)$$

Then, it follows by the Burkholder-Davis-Gundy inequality, Jensen's inequality, Fubini's theorem and the moment bounds for the χ^2 and Gaussian random variables that there is a $C_{r,t} > 0$ such that

$$E \left[\left| \int_0^t V_s^{\frac{1}{2}} dB_s \right|^r \right] \leq C_{r,t} E \left[\int_0^t |V_s|^{\frac{r}{2}} ds \right] < \infty$$

for any $r \geq 2$, $t > 0$ and $\int_0^t V_s^{\frac{1}{2}} dB_s$ is an L^r -martingale for any $r > 0$.

Remark 7. *One can apply Itô's formula to (2.3) and (2.4) to verify they do indeed satisfy (1.5). Hence, one could have just guessed this solution and then checked it. However, nobody every has and it took the development in the appendix for the author to formulate this solution.*

Noting that mathematical models are just approximations of reality, one can sometimes justify picking a Heston model such that Condition (C) is true. We demonstrate simulation for this case in the next section. However, we also want a solution for other parameters not just those satisfying Condition (C). With this in mind, we first define the *Closest Explicit Heston* case:

$$d \begin{pmatrix} \hat{S}_t \\ \hat{V}_t \end{pmatrix} = \begin{pmatrix} \mu_\kappa \hat{S}_t \\ \nu_\kappa - \varrho \hat{V}_t \end{pmatrix} dt + \begin{pmatrix} \sqrt{1 - \rho^2} \hat{S}_t \hat{V}_t^{\frac{1}{2}} & \rho \hat{S}_t \hat{V}_t^{\frac{1}{2}} \\ 0 & \kappa \hat{V}_t^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} dB_t \\ d\hat{\beta}_t \end{pmatrix}, \quad (2.7)$$

$$\text{where } n = \left\lfloor \frac{4\nu}{\kappa^2} + \frac{1}{2} \right\rfloor \vee 1, \quad \nu_\kappa = \frac{n\kappa^2}{4}, \quad \mu_\kappa = \mu + \frac{\rho}{\kappa} (\nu_\kappa - \nu), \quad (2.8)$$

where Condition (C) is valid (with $\nu = \nu_\kappa$). Then, we re-weight the outcomes of the closest explicit Heston to get general Heston solutions.

Remark 8. *Finding the closest explicit Heston solution amounts to selecting n .*

The general Heston model (1.5) without Condition (C) also has an explicit weak solution with respect to some new probability until the volatility drops too low.

Theorem 2. *Let $\varepsilon \in (0, 1)$, $T > 0$, $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \in [0, T]}, P)$ be a filtered probability space, V_0, S_0 be given random variables with $V_0 > \varepsilon$, $\{W^1, \dots, W^n, B\}$ be independent standard Brownian motions with respect to $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \in [0, T]}, P)$,*

$$\hat{S}_t = S_0 \exp \left(\sqrt{1 - \rho^2} \int_0^t \hat{V}_s^{\frac{1}{2}} dB_s + \left[\mu - \frac{\nu \rho}{\kappa} \right] t + \left[\frac{\rho \varrho}{\kappa} - \frac{1}{2} \right] \int_0^t \hat{V}_s ds + \frac{\rho}{\kappa} (\hat{V}_t - \hat{V}_0) \right) \quad (2.9)$$

$$\hat{V}_t = \sum_{i=1}^n (Y_t^i)^2, \quad \eta_\varepsilon = \inf \left\{ t : \hat{V}_t \leq \varepsilon \right\} \quad \text{and} \quad (2.10)$$

$$\hat{L}_t = \exp \left\{ \frac{\nu - \nu_\kappa}{\kappa^2} \left[\ln(\hat{V}_t) - \ln(\hat{V}_0) + \int_0^t \frac{\kappa^2 - \nu_\kappa - \nu}{2\hat{V}_s} + \varrho ds \right] \right\}, \quad (2.11)$$

where $Y_t^i = \frac{\kappa}{2} \int_0^t e^{-\frac{\kappa}{2}(t-u)} dW_u^i + e^{-\frac{\kappa}{2}t} Y_0^i$ for $i = 1, 2, \dots, n$. Define

$$\beta_t = \sum_{i=1}^n \int_0^t \frac{Y_u^i}{\sqrt{\sum_{j=1}^n (Y_u^j)^2}} dW_u^i + \int_0^{t \wedge \eta_\varepsilon} \frac{\nu - \nu_\kappa}{\kappa \hat{V}_s^{\frac{1}{2}}} ds, \quad \text{and} \quad (2.12)$$

$$\hat{P}(A) = E[1_A \hat{L}_{T \wedge \eta_\varepsilon}] \quad \forall A \in \mathcal{F}_T. \quad (2.13)$$

Then, η_ε is a stopping time and $\hat{L}_{t \wedge \eta_\varepsilon}$ is a L^r -martingale with respect to P for any $r > 0$. Moreover, (B, β) are independent standard Brownian motions and

$$d \begin{pmatrix} \hat{S}_t \\ \hat{V}_t \end{pmatrix} = \begin{cases} \begin{pmatrix} \mu \hat{S}_t \\ \nu - \varrho \hat{V}_t \end{pmatrix} dt + \begin{pmatrix} \sqrt{1 - \rho^2} \hat{S}_t \hat{V}_t^{\frac{1}{2}} & \rho \hat{S}_t \hat{V}_t^{\frac{1}{2}} \\ 0 & \kappa \hat{V}_t^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} dB_t \\ d\beta_t \end{pmatrix}, & t \leq \eta_\varepsilon \\ \begin{pmatrix} \mu_\kappa \hat{S}_t \\ \nu_\kappa - \varrho \hat{V}_t \end{pmatrix} dt + \begin{pmatrix} \sqrt{1 - \rho^2} \hat{S}_t \hat{V}_t^{\frac{1}{2}} & \rho \hat{S}_t \hat{V}_t^{\frac{1}{2}} \\ 0 & \kappa \hat{V}_t^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} dB_t \\ d\beta_t \end{pmatrix}, & t > \eta_\varepsilon \end{cases} \quad (2.14)$$

on $[0, T]$ with respect to \hat{P} .

Proof. See Appendix. \square

Notation: We are using \hat{S}, \hat{V} for solutions to the *closest explicit* Heston model, reserving S, V for the general case. Henceforth, we will use $\hat{\beta}_t = \sum_{i=1}^n \int_0^t \frac{Y_u^i}{\sqrt{\sum_{j=1}^n (Y_u^j)^2}} dW_u^i$ and $\beta_t = \hat{\beta}_t + \int_0^{t \wedge \eta_\varepsilon} \frac{\nu - \nu_\kappa}{\kappa \hat{V}_s^{\frac{1}{2}}} ds$.

Remark 9. *With respect to the manufactured measure \hat{P} , (\hat{S}_t, \hat{V}_t) satisfies the general Heston model (1.5) until the stopping time η_ε and then the Closest Explicit Heston model (2.7) after that. Conversely, since $\mu - \frac{\nu \rho}{\kappa} = \mu_\kappa - \frac{\nu_\kappa \rho}{\kappa}$, we find that (\hat{S}, \hat{V}) satisfies (2.7) for all $t \in [0, T]$ with respect to P by (2.9) and Theorem 1.*

Our first concern about Theorem 2 is: The desired solution here is only good until η_ε , i.e. until the volatility drops too low (or we hit the final ‘simulation time’ T). From a finance viewpoint, one can ask: “Is it realistic that the volatility of

my asset drops to zero any way?”. Usually, this constraint of not being able to simulate through essentially deterministic price change is not a practical issue and, even if this happens, we just fall back to the closest explicit alternative. Our second concern is: The desired solution here is with respect to a manufactured probability \hat{P} . However,

- (1) this manufactured-probability solution is ideal for option pricing calculations,
- (2) this manufactured-probability solution is also excellent for pricing derivatives via Monte-Carlo-type simulation.

To illustrate the last point, we suppose we have independent copies $\{(\hat{S}^j, \hat{V}^j, \hat{L}^j)\}_{j=1}^N$ of $(\hat{S}, \hat{V}, \hat{L})$ and $\eta_\varepsilon^j = \inf \left\{ t : \hat{V}_t^j \leq \varepsilon \right\}$. Then, using the law of large numbers (for weakly-dependent variables) and \hat{L} 's martingale property

$$\begin{aligned} \frac{1}{N} \sum_{j=1}^N \hat{L}_{t \wedge \eta_\varepsilon^j}^j g(\hat{S}_{[0,t]}^j, \hat{V}_{[0,t]}^j, \Delta_{[0,t]}^N) &\rightarrow E[\hat{L}_{t \wedge \eta_\varepsilon} g(\hat{S}_{[0,t]}, \hat{V}_{[0,t]}, \Delta_{[0,t]})] \\ &= \hat{E}[g(\hat{S}_{[0,t]}, \hat{V}_{[0,t]}, \Delta_{[0,t]})] \end{aligned} \quad (2.15)$$

for any bounded, measurable function g and $t \leq T$, where \hat{E} denotes expectation with respect to \hat{P} , $\Delta_{[0,t]}^N$ is the empirical process $\frac{1}{N} \sum_{j=1}^N \delta_{\hat{L}_{[0,t]}^j, \hat{S}_{[0,t]}^j, \hat{V}_{[0,t]}^j}$ and $\Delta_{[0,t]}$ is the joint distribution of $(\hat{L}_{[0,t]}, \hat{S}_{[0,t]}, \hat{V}_{[0,t]})$. (Here, $\hat{L}_{[0,t]}, \hat{S}_{[0,t]}, \hat{V}_{[0,t]}$ denote the paths of $\hat{L}, \hat{S}, \hat{V}$ over $[0, T]$ held constant after t .) (2.15) is what we need for (SLLN-A, SLLN-b) and therefore to use $\{(\hat{S}^j, \hat{V}^j, \hat{L}^j)\}_{j=1}^N$ in our SA Pricing Algorithm of the previous subsection. In the next subsection, we reduce these theorems to useful algorithms that can be used for simulation or within the LSM and SA option-pricing algorithms.

Example 1. For pricing an American call option with strike price K , we would use $g(\hat{S}_{[0,t]}^j, \hat{V}_{[0,t]}^j, \Delta_{[0,t]}^N) = e^{-\mu \tau_0^{J,j}} (\hat{S}_{\tau_0^{J,j}}^j - K) \vee 0$, where $\tau_0^{J,j}$ satisfies (1.3) in the LSM algorithm or a similar formula (with slightly different but still asymptotically consistent coefficients $\alpha_t^{J,N}$) in the SA algorithm. Since $\tau_0^{J,j}$ depends upon the paths of \hat{S} and \hat{V} so does $g(\hat{S}_{[0,t]}^j, \hat{V}_{[0,t]}^j, \Delta_{[0,t]}^N)$ in American (and Asian) option pricing examples. Since $\tau_0^{J,j}$ uses projection estimates that depend on the other particles, we have to include the empirical process $\Delta_{[0,t]}^N$, which results in weakly interacting variables instead of independent ones. To justify the weakly-interacting SLLN in this example, we note from previous discussion that the projection estimates converge to the desired projection, which no longer depends upon the other particles. Also, the exact dependence of $\tau_0^{J,j}$ on the other particles and the paths is not critical but can be determined from the SA algorithm and the Weighted Heston Algorithm to follow.

2.3. Weighted and Explicit Heston Simulation.

Defining constants

$$a = \sqrt{1 - \rho^2}, \quad b = \mu - \frac{\nu \rho}{\kappa}, \quad c = \frac{\rho \varrho}{\kappa} - \frac{1}{2}, \quad d = \frac{\rho}{\kappa}, \quad e = \frac{\nu - \nu_\kappa}{\kappa^2}, \quad f = e \frac{\kappa^2 - \nu - \nu_\kappa}{2},$$

we find that (2.9,2.11) can be rewritten as

$$\widehat{S}_t = \widehat{S}_{t-1} \exp \left(a \int_{t-1}^t \widehat{V}_s^{\frac{1}{2}} dB_s + b + c \int_{t-1}^t \widehat{V}_s ds + d (\widehat{V}_t - \widehat{V}_{t-1}) \right) \quad (2.16)$$

$$\widehat{L}_t = \widehat{L}_{t-1} \exp \left\{ e \left(\ln \left(\frac{\widehat{V}_t}{\widehat{V}_{t-1}} \right) + \varrho \right) + f \int_{t-1}^t \frac{1}{\widehat{V}_s} ds \right\}. \quad (2.17)$$

The stochastic integral in (2.16) is conditionally (given \widehat{V}) Gaussian since \widehat{V} and B are independent so simulation is just a centered normal random variable with variance $a^2 \int_{t-1}^t \widehat{V}_s ds$. Even the weight (2.17) avoids stochastic integrals. There are a number of choices for the two deterministic integrals to be computed like:

$$\begin{aligned} \text{Trapezoidal: } \int_{t-1}^t \widehat{V}_s ds &\approx \frac{1}{2M} \left\{ \widehat{V}_{t-1} + \widehat{V}_t + 2 \sum_{l=1}^{M-1} \widehat{V}_{t-\frac{l}{M}} \right\} \\ \text{Simpson's } \frac{1}{3}: \int_{t-1}^t \widehat{V}_s ds &\approx \frac{1}{3M} \left\{ \widehat{V}_{t-1} + \widehat{V}_t + 2 \sum_{l=1}^{\frac{M}{2}-1} \widehat{V}_{t-\frac{2l}{M}} + 4 \sum_{l=1}^{\frac{M}{2}} \widehat{V}_{t-\frac{2l-1}{M}} \right\} \\ \text{Simpson's } \frac{3}{8}: \int_{t-1}^t \widehat{V}_s ds &\approx \frac{3}{8M} \left\{ \widehat{V}_{t-1} + \widehat{V}_t + 2 \sum_{l=1}^{\frac{M}{3}-1} \widehat{V}_{t-\frac{3l}{M}} + 3 \sum_{l=1}^{\frac{M}{3}} \widehat{V}_{t-\frac{3l-2}{M}} + 3 \sum_{l=1}^{\frac{M}{3}} \widehat{V}_{t-\frac{3l-1}{M}} \right\} \end{aligned}$$

and similar formulae for $\int_{t-1}^t \frac{1}{\widehat{V}_s} ds$. Naturally, all of these will converge to the integral as $M \rightarrow \infty$. V does not satisfy the necessary smoothness conditions for the classical errors of these numeric integral methods so it is unknown which will perform better. Indeed, simulations will show there is very little difference on our examples. Finally, it will be notationally convenient to restrict to the case n is even (the odd case is a minor modification) and to define three more constants

$$\sigma = \kappa \sqrt{\frac{1 - e^{-\frac{\varrho}{2}}}{4\varrho}}, \quad \alpha = e^{-\frac{\varrho}{4}} \text{ and } n_2 = \frac{n}{2}.$$

The algorithm (with the hats removed for notational ease) is now as follows:

Initialize: $\{(S_0^j, L_0^j, \eta_\varepsilon^j) = (S_0, 1, T)\}_{j=1}^N, \{Y_0^{j,i} = \sqrt{\frac{V_0}{n}}\}_{j,i=1}^{N,n}$.

Repeat: for times $t = 1, 2, \dots, T$ do

Repeat: for particles $j = 1, 2, \dots, N$ do

(1) $V_{t-\frac{1}{2}}^j = 0, V_t^j = 0$

(2) **Repeat:** for $i = 1, 2, \dots, n_2$ do

(a) Draw $[0, 1]$ -uniform U_1, U_2, U_3, U_4

(b) $Y_{t-\frac{1}{2}}^{j,2i-1} = \alpha Y_{t-1}^{j,2i-1} + \sigma \sqrt{-2 \log U_1} \cos(2\pi U_2)$ (Use Box-Meuller for normals)

- (c) $Y_{t-\frac{1}{2}}^{j,2i} = \alpha Y_{t-1}^{j,2i} + \sigma \sqrt{-2 \log U_1} \sin(2\pi U_2)$
- (d) $Y_t^{j,2i-1} = \alpha Y_{t-\frac{1}{2}}^{j,2i-1} + \sigma \sqrt{-2 \log U_3} \cos(2\pi U_4)$
- (e) $Y_t^{j,2i} = \alpha Y_{t-\frac{1}{2}}^{j,2i} + \sigma \sqrt{-2 \log U_3} \sin(2\pi U_4)$
- (f) $V_{t-\frac{1}{2}}^j = V_{t-\frac{1}{2}}^j + (Y_{t-\frac{1}{2}}^{j,2i-1})^2 + (Y_{t-\frac{1}{2}}^{j,2i})^2$, $V_t^j = V_t^j + (Y_t^{j,2i-1})^2 + (Y_t^{j,2i})^2$
- (3) Set $IntV^j = \frac{V_{t-1}^j + 4V_{t-\frac{1}{2}}^j + V_t^j}{6}$ (Simpson's $\frac{1}{3}$ rule, $M = 2$)
- (4) Set $N^j = \mathcal{N}\left(0, a\sqrt{IntV^j}\right)$ (centered normal RV)
- (5) $S_t^j = S_{t-1}^j \exp(N^j + b + c IntV^j + d (V_t^j - V_{t-1}^j))$
- (6) $Z_t^j = p(t, S_t^j)$ (Discounted Payoff e.g. $e^{-\mu t}(K - S_t^j) \vee 0$ for American put)
- (7) If $t \leq \eta_\varepsilon^j$ then
 If $V_{t-\frac{1}{2}}^j \wedge V_t^j > \varepsilon$ then $L_t^j = L_{t-1}^j \exp\left\{e\left(\ln\left(\frac{V_t^j}{V_{t-1}^j}\right) + \varrho\right) + \frac{f}{6}\left[\frac{1}{V_{t-1}^j} + \frac{4}{V_{t-\frac{1}{2}}^j} + \frac{1}{V_t^j}\right]\right\}$
 Otherwise $\eta_\varepsilon^j = t - 1$

Remark 10. *There are some practical notes about using this algorithm:*

- (1) $e^{-\mu}$ is the discount factor in (6) so $e^{\mu t}$ dollars at time t are considered as valuable as \$1 at time 0.
- (2) To price Asian options, where our payoff is in terms of the running average price not the spot price, on the Heston model we initiate $R_0 = 0$, add a step:
 (5a) $R_t^j = \frac{t-1}{t} R_{t-1}^j + \frac{1}{t} S_t^j$
 and change the payoff process in (6) to $Z_t^j = p(t, R_t^j)$. You can then impose a “lockout period” by resetting the Z_t^j to 0 for those times.
- (3) In the Theorem 1 case of $\nu = \frac{\kappa^2}{4}$, we have explicit solutions without the need of weights. In this case, we can skip Step (7) and remove all references to η_ε and L^j in this algorithm. We call this reduced algorithm for Theorem 1 the Explicit Heston Simulation algorithm and the general algorithm (as stated above) for Theorem 2 the Weighted Heston Simulation algorithm.
- (4) For added efficiency, Box-Meuller could be used in Step (4) as well. Moreover, you could lump constants together to reduce multiplications (at the cost of code readability). We do not employ these added efficiencies herein.
- (5) A larger M or a better integral approximation could also be used to improve performance in Step (3). We used $M = 2$ and Simpson's $\frac{1}{3}$ rule for algorithm clarity reasons only.

To understand the need to stop (at η_ε) before the volatility gets too small, we consider the situation where the volatility $V_t^{\frac{1}{2}} = 0$. Then, the (closest explicit and general) Heston volatility equations become deterministic

$$d\widehat{V}_t = \nu_\kappa dt, \quad dV_t = \nu dt$$

and it is obvious which solution one has. This makes model distributions singular to each other when $\nu_k \neq \nu$.

3. PERFORMANCE OF EXPLICIT SOLUTION SIMULATION

We compare our algorithms numerically to some of the more popular methods, first in this section on simulation and then in the next section on progressively more involved option pricing problems. All experiments in both sections are performed on the same computer system, consisting of a Lenovo X240s Laptop with a 4th generation Intel Core i7-4500U @ 1.80GHz processor, 8GB DIMML memory, 1TB 5400 RPM hard disk, Windows 8.1 64 bit operating system and the C++ compiler from Visual Studio professional 2013.

3.1. Non-failure of Explicit Heston Simulation. We will call a simulation where a *negative* volatility is produced a *failure* and the first time this occurs is defined as the *break time* τ . The Euler and Milstein methods both fail by producing negative volatility values that can not be square rooted without change (like setting to zero). Conversely, our Explicit Heston algorithm can not fail in this manner as the volatility is exact and stays non-negative by its construction.

First, suppose $\mu = 0.0319, \rho = -0.7, \varrho = 6.21, \kappa = 0.61$ and $\nu = \frac{\kappa^2}{4}$ so the (SDE model) volatility can hit zero but can not go negative. Our initial state is $S_0 = 100, V_0 = 0.010201$ and we run the simulation either 10,000 or 40,000 times until final time $T = 50$. We use either 100 or 200 discretization steps between each integer time. The relative breaking frequency of Euler and Milstein simulations are shown in Tables 1 below.

Scheme	Euler		Milstein	
N	10,000	40,000	10,000	40,000
Steps	100	200	100	200
$\tau = 1$	0.972386	0.972184	0.932158	0.914071
$\tau = 2$	0.026434	0.025734	0.062245	0.077341
$\tau = 3$	0.001134	0.001033	0.005166	0.007731
$\tau = 4$	0.000045	0.0000465	0.000394	0.000777
$\tau = 5$	0.000001	0.0000025	0.000037	0.0000713
$\tau = 50$	0	0	0	0

TABLE 1. Relative breaking frequency for $\nu = \frac{\kappa^2}{4}, \kappa = 0.61, \varrho = 6.21$

Ideally, there should not be any failures, so every simulation should reach $\tau = T = 50$ but actually none do. One might think that this only happens when the volatility is supposed to hit zero. However, increasing ν to $\frac{\kappa^2}{2}$, which is the critical or first case that the volatility should not hit 0, we still encounter the same problem, especially for the Euler scheme.

Scheme	Euler		Milstein	
N	10,000	40,000	10,000	40,000
Steps	100	200	100	200
$\tau = 1$	0.802964	0.767827	0.000492	0
$\tau = 2$	0.147584	0.165	0.000488	0
$\tau = 3$	0.037084	0.047847	0.000506	0
$\tau = 4$	0.009277	0.013768	0.000524	0
$\tau = 5$	0.002313	0.003941	0.000484	0
$\tau = 50$	0	0	0.976822	1

TABLE 2. Relative breaking frequency for $\nu = \frac{\kappa^2}{2}$, $\kappa = 0.61$, $\varrho = 6.21$

For $\nu = \frac{\kappa^2}{2}$, we see that Milstein scheme with 200 steps works well while the Euler scheme volatility still goes negative in every simulation.

3.2. Comparison of Explicit Heston Simulation. We provide an example of our Explicit Heston simulation and compare this to the traditional Euler and Milstein simulation methods. In this approach, we create a *ground truth* to judge performance from by fixing Brownian paths B, β and running the Milstein method once with the ridiculously small time step $\Delta t = 1/2,000$. We then used these fixed B, β paths to calculate the error in the simulations discussed in this subsection. To get *time* estimates we resort back to the normal efficient algorithms that would be used in practice. In this manner, we obtain comparable *path-by-path* simulation error with execution time estimates for the typical time it would take to produce those errors.

For this example, we used the following collection of parameters: $\nu = \nu_\kappa = \frac{\kappa^2}{4}$, $\mu = 0.0319$, $\rho = -0.7$, $\varrho = 6.21$, $\kappa = 0.61$ and $T = 10$. We also take the (non-ground-truth) Euler and Milstein time steps to be $\Delta t = 1/M$, where the number of steps are $M = 200, 400, 1,000$. Since Condition (C) holds we can remove all reference to L and η from the previously-given Heston simulation algorithm. Tables 3 and 4 below show the performance and execution time of our Explicit Heston algorithm with the Trapezoidal, Simpson's $\frac{1}{3}$ as well as Simpson's $\frac{3}{8}$ rule along with the Euler and Milstein methods. For clarity, the performance is defined in terms of RMS error. The RMS error for the Milstein method is:

$$E^M = \sqrt{\frac{1}{N} \sum_{t=1}^T \sum_{i=1}^N \left[(S_t^{M,i} - S_t^i)^2 + (V_t^{M,i} - V_t^i)^2 \right]},$$

with S^M, V^M being the price and volatility using the Milstein method and S, V being the ground truth price and volatility. The other RMS errors are defined similarly.

	Euler Scheme			Milstein Scheme		
Steps	200	400	1,000	200	400	1,000
RMS	18.8256	14.1382	9.79565	10.5435	7.08773	4.2306
Time	0.81	1.672	4.026	0.936	1.733	4.731

TABLE 3. Comparison of Accuracy and Execution Time

	Explicit Solution			
	Trapezoidal		Simpson's $\frac{1}{3}$	Simpson's $\frac{3}{8}$
M	1	6	6	6
RMS	3.62901	2.89821	2.91712	3.08562
Time	0.0054	0.012	0.01	0.014

TABLE 4. Accuracy and Execution Time for Explicit Solution Simulation

It is clear that our Explicit Heston method is more accurate and quicker than the other methods. However, to get a single measure of improvement, we combine performance and time factors and define

$$\text{Explicit Gain} = \frac{\tau_{\text{Other}}}{\tau_{\text{Explicit}}}, \quad (3.1)$$

where τ_{Explicit} and τ_{Other} are the execution times for our Explicit Heston algorithm and some other method for a *fixed* performance. However, it is very hard to get the Milstein method, let alone the Euler one, to perform as well as the worst we can do with the explicit weak solution method so we plot existing Milstein points and extend a smooth curve to get some estimates. (Part of the difficulty of collecting Milstein data with more steps here is that we would have to re-run the ground truth with a much higher number of steps, which would exceed our computational limits.) In this way, we estimate it would take Milstein at least 5.9 s with a very high number of steps to match the Explicit's 3.62901 RMS so the explicit gain in execution time would be 1093. We follow a similar procedure for Euler and tabulate the gains in Table 5.

Method	Euler	Milstein
Explicit Gain Over	2630	1093

TABLE 5. Explicit Gain over Euler and Milstein

Clearly there is significant gain in using our Explicit simulations. There are similar gains (exceeding 1000) at other error levels and durations T .

4. PERFORMANCE OF SA AND HESTON ALGORITHMS

Now, we turn our attention to option pricing. For simplicity, we will use the same bases functions for volatility, price and, in the case of Asian options, average price. This means we will use $J = j^2$ (or $J = j^3$ in the case of Asian options) functions of the form $e(s, v) = e_{k_1}(s)e_{k_2}(v)$ for $k_1, k_2 \in \{1, \dots, j\}$. Moreover, since there was little difference between Trapezoidal, Simpson's $\frac{1}{3}$ and Simpson's $\frac{3}{8}$ in the simulation experiment above, we will only consider the Trapezoidal method within our Heston algorithms to follow.

4.1. Weighted Heston on American Puts with LSM Algorithm. First, we compare our Weighted Heston algorithm with the traditional Euler and Milstein methods in pricing an American put option. It was shown in the previous section that Explicit Heston simulation is three orders of magnitude faster (for the same accuracy) as Euler and Milstein simulation. Now, we consider the real problem of option pricing and answer the question: “Does much does faster simulation translate into significantly faster option pricing where, in addition to simulation, one has to do dynamic programming to price?” In addition, we do not assume the explicit case where Condition (C) holds, which means the likelihoods must be computed. For clarity, we do not use our SA algorithm yet, but rather stick to the LSM algorithm. We simply substitute our Weighted Heston as well as the other methods into the simulation portion of this popular algorithm.

We use Heston and American put option parameters: $\nu = \frac{8.1\kappa^2}{4}, \mu = 0.0319, \rho = -0.7, \varrho = 6.21, \kappa = 0.2, S_0 = 100, V_0 = 0.501, T = 50$ and the strike price $K = 100$. Here $n = 8.1 \notin \mathbb{N}$ and Condition (C) does not hold. Hence, we use the full Weighted Heston algorithm with $\nu_\kappa = 2\kappa^2$ in the closest explicit Heston model. Finally, we use the weighted Laguerre polynomials

$$\begin{aligned} e_1(x) &= L_0(x) = \exp(-x/2) \\ e_2(x) &= L_1(x) = \exp(-x/2)(1 - x) \\ e_3(x) &= L_2(x) = \exp(-x/2)(1 - 2x + \frac{x^2}{2}) \\ e_j(x) &= L_{j-1}(x) = \exp(-x/2) \frac{e^x}{(j-1)n!} \frac{d^j}{dx^j} (x^j e^{-x}) \end{aligned}$$

with $j = 3, J = 3^2$ for the LSM pricing process.

Pre-experiments show that all these methods work and converge to the same *nearly* correct answer as the number of particles increases and the step size decreases. The fact that they do not converge to the correct answer is due to the finiteness of the collection of functions $\{e_k\}_{k=1}^J$ used. Hence for a ground truth, we run the LSM algorithm with Milstein simulation with extraordinarily fine time step and an enormous number of particles but still for small $j = 3$ (so the LSM algorithm can even work). (We will get around this small j issue later when using SA instead of LSM.) Table 6 gives the ground truth using a million particles with $\Delta t = 1/M = 1/1,000$.

Ground Truth	
N	1, 000, 000
M	1, 000
Option Price	12.269

TABLE 6. Ground Truth of the American Put Price

To compare performance, we will fix the error for the three methods and compare their execution time. The error is defined as:

$$error = \frac{1}{\# \text{ Seeds}} \sum_{\text{Seeds}} |P^E - P| \quad (4.1)$$

with P^E being the option price obtained by running N particles with Euler scheme and P being the ground truth option price (except $J = 3^2$ still). The other error are defined similarly. The results are provided in Tables 7 and 8 for the cases where we can tolerate a pricing error of 4 and 3 cents respectively.

	Euler	Milstein	Weighted Heston
N	10, 000	7, 225	2, 500
M	100	85	15
Price	12.3116	12.2254	12.2258
Error	0.0426	0.0436	0.0432
Time	17.4178	13.156	1.387
Time Gain	1	1.324	12.562

TABLE 7. American Put Execution Time - Low Accuracy case

	Euler	Milstein	Weighted Heston
N	40, 000	30, 625	3, 500
M	200	175	17
Price	12.3013	12.2367	12.2366
Error	0.0323	0.0323	0.0324
Time	143.356	84.6254	2.20966
Time Gain	1	1.694	64.877

TABLE 8. American Put Execution Time - High Accuracy case

In Tables 7 and 8, we defined a

$$\text{Time Gain} = \frac{\tau_{\text{LSM-Euler}}}{\tau_{\text{Other}}}, \quad (4.2)$$

where $\tau_{\text{LSM-Euler}}$ is the time required to achieve a specified accuracy using the LSM algorithm with Euler simulation and τ_{Other} is the time required to obtain the same level of accuracy with some other method. This resembles the Explicit Gain in

(3.1). Since in this experiment only the LSM is used, Time Gain here describes how many times faster option pricing with the Milstein and Weighted Heston algorithms are than the basic Euler Scheme with the same error. As presented above, the weighted Heston algorithm shows a remarkable improvement over the traditional discretization method. The speed advantage is more significant when we require a higher accuracy. Later, we will replace the LSM with the SA algorithm to increase speed further and to enjoy the higher accuracy afforded by larger J .

4.2. Weighted Heston on Asian Straddles with LSM Algorithm. We compare Euler, Milstein and our weighted Heston by pricing Asian Straddles via the LSM algorithm. The discounted payoff process for an Asian straddle is $Z_t = e^{-\mu t}|R_t - K|$, where R is the running average of the Heston price, calculated as

$$R_t = \frac{t-1}{t}R_{t-1} + \frac{1}{t}S_t, \quad (4.3)$$

and K is the strike price. As the Asian Straddles option pricing model is a three factor model (spot price, average price and volatility), we will only use $j = 2$ for each factor for computational reasons. The other parameters remain the same as the American put option: $\nu = \frac{8.1\kappa^2}{4}$, $\mu = 0.0319$, $\rho = -0.7$, $\varrho = 6.21$, $\kappa = 0.2$, $S_0 = 100$, $V_0 = 0.501$, $T = 50$ and the strike price $K = 100$. The groundtruth of the Asian Straddles price, computed by Milstein's method with a million particles and a very fine time step, is used for measuring the error and is given in Table 9.

Ground Truth	
N	1,000,000
M	1,000
Option Price	136.174

TABLE 9. Ground Truth of the Asian Straddle Price

The Asian straddle time gains, given in Tables 10 and 11 (to follow), also indicate the efficiency of the weighted Heston as it did for the American put.

	Euler	Milstein	Weighted Heston
N	10,000	4,900	3,510
M	100	70	12
Price	135.956	135.952	136.019
Error	0.218	0.214	0.222
Time	18.8237	11.2313	1.8943
Time Gain	1	1.676	9.937

TABLE 10. Asian Straddle Execution Time - Low Accuracy case

For lower accuracy, the weighted Heston performs about ten times as fast as the traditional method with the fixed error. As with the American put, this outperformance improves as one desires higher accuracy.

	Euler	Milstein	Weighted Heston
N	40,000	25,600	4,800
M	200	160	13
Price	136.043	136.046	136.303
Error	0.131	0.128	0.124
Time	145.864	73.958	2.861
Time Gain	1	1.972	50.984

TABLE 11. Asian Straddle Execution Time - High Accuracy case

Our weighted Heston method shows a rather strong performance in the high accuracy case since the Time Gain increases to around 51, which means we can get the same accuracy with $\frac{1}{51}$ the execution time. Indeed, these results show that the simulation component of the LSM algorithm is very important and that our Weighted Heston method is the best method.

We can speculate on the reason the outperformance is less for the Asian straddle than the American put: The method and time in going from spot price to running average price is the same, whether we use Euler, Milstein or Weighted Heston. Moreover, adding a constant (running average price time) to the numerator and denominator of (4.2) will drag the Time Gain ratio towards 1.

4.3. Comparison of SA and LSM on American Puts. Having shown that our Explicit and Weighted Heston simulation methods can be superior to the Euler and Milstein methods in option pricing, we turn our attention to comparing the SA and LSM algorithms with different numbers and types of functions $\{e_k\}_{k=1}^J$ used. In this subsection, we will use model parameters: $\mu = 0.0319, \rho = -0.7, \varrho = 6.21, \kappa = 0.61, K = 100, S_0 = 100, V_0 = 0.0102, T = 50$ and $\nu = \frac{1}{2}\kappa^2$ so the Explicit algorithm applies. We use $\gamma = 2.115, 0.195, 0.0095$ for $J = 2^2, 3^2, 4^2$ respectively in the case $N = 10,000$ and $\gamma = 1.068, 0.762, 0.0082$ for $J = 2^2, 3^2, 4^2$ respectively in the case $N = 100,000$ below as these were determined numerically to be reasonable choices. All the prices are calculated by taking the average of 100 independent experiment.

First, we show that the LSM algorithm can fail numerically when adding more weighted Laguerre functions in an attempt to achieve higher price accuracy. Tables 12, 13 show this along with performance.

	SA Price	SA Time	LSM Price	LSM Time
J=2²	8.44858	0.11298	8.40775	0.124679
J=4²	8.49936	0.14411	8.38028	0.258755
J=8²	8.41892	0.2566856	5.58625	2.13897

TABLE 12. SA and LSM with $N = 10,000$

	SA Price	SA Time	LSM Price	LSM Time
$J=2^2$	8.4213	1.24712	8.39404	1.51143
$J=4^2$	8.50788	1.79924	8.51376	2.7524
$J=8^2$	8.51644	2.64996	7.18587	20.1488

TABLE 13. SA and LSM with $N = 100,000$

We can draw several conclusions from Tables 12 and 13. First, there is a large execution time advantage for our SA algorithm over the popular LSM algorithm, especially as J increases and matrix inversion becomes difficult. For small numbers of the basis functions, SA is about 10% faster than LSM. However, when the number of basis functions increases, the SA time performance becomes even more superior. For example, when $J = 8^2$, the SA algorithm is nearly ten times faster, yet much more accurate. Next, given *enough* particles (eg. $N = 100,000$ here), prices and pricing accuracy should both increase as we add more basis functions because we will obtain a better estimate of the optimal stopping time. Table 13 does demonstrate that as J increases from 2^2 to 8^2 the SA option prices increase and the SA algorithm does not break. Indeed, it should never break as it avoids the numeric issues of matrix inversion. The LSM algorithm does break as prices dive and time spikes for large J in both Table 12 and Table 13 due to ill-conditioned matrix inversion in the least squares estimate. Prices fall in Table 12 for the SA algorithm for a different reason: When N is small the projection parameter estimates are often bad, especially when there are a lot of parameters to estimate, and optimal stopping is easily missed, even when J is large. More bad (low N) parameter estimates with larger J is not necessarily an advantage and prices can vary in either direction as you increase J with small N fixed. To provide further evidence of this expected price improvement in J given large enough N and to find the ground truth for pricing, we also run the Stochastic Approximation method with $N = 1,000,000$ and $J = 12^2$. As shown in Table 15, the American put option price rises to 8.58712.

Ground Truth	
N	1,000,000
J	12^2
γ	0.99294
SA Option Price	8.58712

TABLE 14. Ground Truth of the American Put Price using SA method

The SA prices in Tables 12 and 13 were heading in the right direction. The SA algorithm behaves better than the LSM, especially as the desired accuracy increases.

4.4. Comparison of SA and LSM on Asian Calls. We continue our comparison of SA and LSM algorithms but now on an Asian Call option and in a situation where the Weighted Heston has to be used. First an observation: Since we are pricing

options on average spot price in Asian options, which varies less and less as time goes on, the pricing problem should be easy. Suppose we are slightly off on our optimal stopping time and the optimal stopping time is not near the beginning of the period. Then, the average price and the payoff will not differ much between the optimal stopping time and our estimate (due to the averaging) and hence our price estimate and the optimal option price will not either.

In this section, we will use model parameters: $\nu = \frac{8.1\kappa^2}{4}$, $\mu = 0.0319$, $\rho = -0.7$, $\varrho = 6.21$, $\kappa = 0.2$ and $T = 50$ so $n = 8.1$ and $\nu_\kappa = 2\kappa^2$ is used in the Closest Explicit Heston. The ground truth for this experiment is:

Ground Truth	
N	1,000,000
J	12^3
γ	0.962
SA Option Price	31.3455

TABLE 15. Ground Truth of the Asian Call Price using SA

Again, it is impossible to get that accurate on a standard contemporary computer with the LSM method due to matrix inversion issues for large J . Also, Euler and Milstein would not finish within a two week time frame for this value of N and a high enough number of steps M . All the prices are calculated by taking the average of 100 independent experiments.

Following the same procedure as pricing the American Put option, we first consider performance with different numbers of basis functions and show this in Table 16:

	SA Price	SA Time	LSM Price	LSM Time
N	100,000		100,000	
J=2³	31.3411	11.2404	25.2365	12.511
J=4³	31.3411	36.2066	20.3398	92.432

TABLE 16. SA and LSM with $N = 100,000$

For completeness, we used $\gamma = 1, 0.824$ for $J = 2^3, 4^3$ respectively.

We can clearly see that the LSM fails already when $J = 2^3$. The main reason still lies in the matrix inversion part: Since the Asian Calls is a three factor model, we have to invert a 8×8 matrix. Indeed, when you have both price and average price there is a greater chance of this matrix having nearly linearly dependent rows and hence being highly ill-conditioned to inversion.

The SA algorithm does not fail even for large numbers of basis functions. The price remains the same for $J = 2^3$ and 4^3 due to the averaging mentioned in the first paragraph above. Indeed, a comparison between Tables 15 and 16 shows that the SA algorithm with $J = 2^3, 4^3$ and $N = 100,000$ already gives a rather close result to the ground truth.

4.5. Comparison of Weighted-SA and Euler-LSM on American Puts. Our final results are comprehensive, showing the overall gain of the methods suggested herein over the traditional Euler-LSM method. The model parameters used in this section are: $\nu = \frac{8.1\kappa^2}{4}$, $\mu = 0.0319$, $\rho = -0.7$, $\varrho = 6.21$, $\kappa = 0.2$ and $T = 50$ so $n = 8.1 \notin \mathbb{N}$ and Condition (C) does not hold. Hence, we will use the full Weighted Heston algorithm with $\nu_\kappa = 2\kappa^2$ in the closest explicit Heston model. The initial state $S_0 = 100$, $V_0 = 0.102$, and the strike price $K = 100$.

The ground truth price is found using the weighted Heston in SA algorithm with fine meshing. The result is given in Table 17.

Ground Truth	
M	5
N	1,000,000
J	12^2
γ	0.00628
SA Option Price	7.9426

TABLE 17. Optimal American Put Price

We run the actual experiment by varying M, N, J to obtain the option price for fixed execution times.

	E-LSM	W-SA	E-LSM	W-SA
M	100	5	100	5
N	10,000	65,000	10,000	90,000
J	4^2	8^2	5^2	6^2
Price	7.371	7.932	6.944	7.9347
Error	0.572	0.0103	0.9986	0.00788
Time	19.662	19.433	22.702	22.528
Performance Gain	1	55.534	1	126.726

TABLE 18. Performance comparison on American Puts

(For clarity, γ was taken as 0.00096 and 0.013 in the $N = 65,000$ and 90,000 cases respectively.)

The Performance Gain is defined (in a similar way as the time factor in the previous section) to represent the relative accuracy of each method given a fixed computation time. The traditional Euler-LSM method does not fail in $J = 4^2$ case as is shown in the first column. In this situation, the accuracy will be increased by 55 times by switching to the Weighted-SA method. The last two columns present the case that Euler-LSM starts to fail. As we will not know the ground truth, hence if the LSM is failing in practice, it is still resonable to conduct the comparison in this case. We found that the relative accuracy has risen to more than 126 times using the new algorithms, which is an impressive *two-orders of magnitude* improvement

for pricing options in the real market. We mention in future work below ways to increase this even more.

5. CONCLUSIONS AND FUTURE WORK

We can make the following conclusions:

- (1) The Heston model has explicit weak stochastic differential equation solutions. These solutions can be easily constructed when Condition (C) holds. Otherwise, they have an explicit likelihood that can be used either as a weight or to change probabilities so the desired model holds.
- (2) The Explicit Heston algorithm should be considered for simulation when it applies. In particular, it does not produce negative volatility values and it compares favourably in terms of both performance and execution time to the Euler and Milstein methods. Indeed, we showed a three order of magnitude overall advantage.
- (3) The Weighted (or Explicit when it applies) Heston algorithm should be considered for Monte Carlo option prices. It compares favorably to the Euler and Milstein methods on the American and Asian option pricing examples considered herein. (It is also much easier to implement than the Broadie-Kaya method on path-dependent options.)
- (4) Stochastic Approximation (SA) should be considered as a favorable alternative to Least-Squares regression in the LSM algorithm. It avoids numerically nasty matrix inversion and thereby allows a larger number J of functions in the projection and closer approximation of the future payoff conditional expectations.

Potential future work includes:

- (1) The SA pricing algorithm should be explored more. Are the situations where the LSM algorithm should still be used? Will other stochastic approximation schemes yield better performance? Are there any guidelines for selecting the functions (e_k) ?
- (2) The Explicit and Weighted Heston algorithms need to be explored more. What type of numeric integration is best? Are there variations of the algorithm that perform better?
- (3) Resampling could be employed to improve the performance of the Weighted Heston algorithm. Currently, we keep all paths, including those that have very low weight. It may be a better strategy to split the higher weight ones and remove the lower weight ones in an unbiased way. However, this must be done in the correct way since American and Asian option pricing are path dependent problems. It will not be enough to just worry about the current particle states. We will have to consider the whole particle paths.
- (4) Precise conditions for rate of convergence results and the optimal rates should be found for the combined Weighted Heston SA algorithm. This is not necessarily simple because of the weak interaction and the path-dependence.

- (5) New explicit weak solutions to other financial models should be investigated. The author is very optimistic that there are explicit three-factor stochastic-mean, stochastic-volatility models for the finding. This would be done along the lines laid out in the appendix.

6. APPENDIX: SOLVING THE SDES

6.1. **Background.** Generally, a weak solution (on a subdomain of \mathbb{R}^p) to

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \quad (6.1)$$

is the triplet of a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, a \mathbb{R}^d -valued Brownian motion $\{W_t, t \geq 0\}$ with respect to $\{\mathcal{F}_t\}_{t \geq 0}$, and an $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted continuous process $\{X_t, t \geq 0\}$ such that (W, X) satisfy Equation (6.1). More restrictively, a strong solution to (6.1) is an $\{\mathcal{F}_t^W\}_{t \geq 0}$ -adapted process X on a probability space (Ω, \mathcal{F}, P) supporting the Brownian motion W , where $\mathcal{F}_t^W \triangleq \sigma\{W_u, u \leq t\}$.

Weak solutions are often handled via martingale problems: Suppose $D \subset \mathbb{R}^p$ is a domain, $C_D[0, \infty)$ denotes the continuous D -valued functions on $[0, \infty)$ with the topology of uniform convergence on compacts, $(L, \mathcal{D}(L))$ is a linear operator on $C(D)$, the continuous \mathbb{R} -valued functions on D , and μ is a probability measure on D . Then, a solution to the $C_D[0, \infty)$ -martingale problem for (L, μ) is any probability measure P_μ on $\Omega = C_D[0, \infty)$ such that the canonical process $\{\omega_t, t \geq 0\}$ satisfies: $P_\mu \omega_0^{-1} = \mu$, and for each $f \in \mathcal{D}(L)$ one has that

$$M_t(f)(\omega) = f(\omega_t) - \int_0^t Lf(\omega_u)du, \quad t \geq 0,$$

is a P_μ -martingale. The martingale problem is well-posed if there is exactly one such probability measure P_μ on $C_D[0, \infty)$.

A weak solution $((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P), \{W_t, t \geq 0\}, \{X_t, t \geq 0\})$ to (6.1) then (see Karatzas and Shreve (1987) p. 317) corresponds to each martingale problem solution P_μ for (L, μ) , with L defined by

$$Lf(x) = \sum_{i=1}^p b_i(x) \partial_{x_i} f(x) + \frac{1}{2} \sum_{i=1}^p \sum_{j=1}^p a_{ij}(x) \partial_{x_i} \partial_{x_j} f(x), \quad (6.2)$$

through the relation $(\Omega, \mathcal{F}) = (C_D[0, \infty), \mathcal{B}(C_D[0, \infty)))$, $X_t = \omega_t$ for $t \geq 0$, $P_\mu = PX^{-1}$, where ω_t denotes the projection function on $C_D[0, \infty)$. $(W_t, \mathcal{F}_t)_{t \geq 0}$ are defined through a martingale representation theorem and $a = \sigma \sigma^T$, where $\sigma \in \mathbb{R}^{p \times d}$. Well-posedness of a martingale problem is with respect to the given operator L (and initial distribution μ). It opens the possibilities of having different sdes with the same operator and hence (under well-posedness) the same law. We will take advantage of this fact in (6.10, 6.11) below.

The Heston model (1.5) corresponds to the martingale problem for operator

$$\begin{aligned} Lf(s, v) = & \mu s \partial_s f(s, v) + (\nu - \rho v) \partial_v f(s, v) + \frac{1}{2} s^2 v \partial_s^2 f(s, v) \\ & + \rho \kappa s v \partial_s \partial_v f(s, v) + \frac{1}{2} \kappa^2 v \partial_v^2 f(s, v). \end{aligned} \quad (6.3)$$

However, b and σ are not bounded nor is $a = \sigma\sigma'$ is strictly positive definite everywhere. Hence, well-posedness of this martingale problem is not immediate. However, it follows from the proofs in Stroock and Varadhan (1969), Stroock and Varadhan (1979) that there is uniqueness up to the first time the volatility hits zero. This means that there is well-posedness in the case $\nu \geq \frac{\kappa^2}{2}$ since it is well known that the (CIR) volatility will not hit zero in this case and we have already discussed existence. As for the remaining case, we mention that others (see Daskalopoulos and Feehan (2011)) have recognized the degenerate nature of the Heston model and considered a different type of existence and uniqueness.

Our work gives explicit construction of the weak solutions that are known to be distributionally unique in the case $\nu \geq \frac{\kappa^2}{2}$. Its importance is in the ability to simulate these explicit constructions. Moreover, our methods may well yield explicit solutions for other financial models.

6.2. Proof of Theorem 1. Stochastic differential equations can be interpreted and solved explicitly either in the strong or weak sense. Weak interpretations are often sufficient in applications like mathematical finance and filtering and allow solutions to a greater number of equations than strong solutions. However, there is also the possibility of finding new explicit strong solutions through the guise of weak solutions, which should not be surprising given the result of Heunis (1986). Moreover, weak solutions can often be converted to (marginals of) strong solutions of a higher dimension sde, which is the first way that we will use weak interpretations. Our approach will be to show everything explicitly in the case $n = 2$ and then explain the necessary changes for $n \in \{1, 3, 4, \dots\}$. However, we first simplify the task by observing the “independently driven” part of the price can be split off.

6.2.1. *Price Splitting.* Suppose that

$$d \begin{pmatrix} S_t^c \\ V_t \end{pmatrix} = \begin{pmatrix} \mu S_t^c \\ \nu - \varrho V_t \end{pmatrix} dt + \begin{pmatrix} \rho S_t^c V_t^{\frac{1}{2}} \\ \kappa V_t^{\frac{1}{2}} \end{pmatrix} d\hat{\beta}_t, \quad (6.4)$$

$$S_t^i = \exp \left(\sqrt{1 - \rho^2} \int_0^t V_s^{\frac{1}{2}} dB_s - \frac{1 - \rho^2}{2} \int_0^t V_s ds \right) \quad (6.5)$$

with respect to independent Brownian motions $\hat{\beta}, B$. Then, it follows by Itô's formula and the independence of $\hat{\beta}, B$ that $S_t = S_t^c S_t^i$ and V_t satisfy (1.5) with $\beta = \hat{\beta}$. Moreover, S^i is conditionally (given V) log-normal and hence trivial to simulate. Hence, we only have to solve (6.4), which we do using weak interpretations to create a higher dimension sde that does satisfy (2.2) and hence has an *explicit* strong solution.

6.2.2. *Volatility in Case $n = 2$.* To ease the notation, we will use Y and Z in place of Y^1, Y^2 in Theorem 1. We consider solutions to a Cox-Ingersoll-Ross (CIR) type Ito equation

$$dV_t = (\nu - \varrho V_t) dt + \kappa \sqrt{V_t} d\hat{\beta}_t, \quad (6.6)$$

for some Brownian motion $\widehat{\beta}$. Let W^1, W^2 be independent Brownian motions so

$$Y_t = \frac{\kappa}{2} \int_0^t e^{-\frac{\rho}{2}(t-u)} dW_u^1 + e^{-\frac{\rho}{2}t} Y_0, \quad Z_t = \frac{\kappa}{2} \int_0^t e^{-\frac{\rho}{2}(t-u)} dW_u^2 + e^{-\frac{\rho}{2}t} Z_0 \quad (6.7)$$

are independent Ornstein-Uhlenbeck processes. It follows by Itô's formula that, if Condition (C) is true (with $n = 2$), then $V = Y^2 + Z^2$ satisfies (6.6) with

$$\widehat{\beta}_t = \int_0^t \frac{Y_u}{\sqrt{Y_u^2 + Z_u^2}} dW_u^1 + \int_0^t \frac{Z_u}{\sqrt{Y_u^2 + Z_u^2}} dW_u^2. \quad (6.8)$$

(Note that $(\widehat{\beta}, W)$ is a standard two dimensional Brownian motion, where $W_t = \int_0^t \frac{Z_u}{\sqrt{Y_u^2 + Z_u^2}} dW_u^1 - \int_0^t \frac{Y_u}{\sqrt{Y_u^2 + Z_u^2}} dW_u^2$, by Levy's characterization.) We call $(V, \widehat{\beta})$ a weak solution since the definition of $\widehat{\beta}$ was part of the solution. V will also be a strong solution if V_t is measurable with respect to $\mathcal{F}_t^{\widehat{\beta}} \doteq \sigma\{\widehat{\beta}_u, u \leq t\}$. A strong solution does not immediately follow from the Yamada-Watanabe theorem since the conditions for pathwise uniqueness in e.g. Theorem IX.3.5 of Revuz and Yor (1999) can not immediately be validated. Moreover, explicit form in terms of only $\widehat{\beta}$ is unknown. (Example 3.4 of Kouritzin (2000) shows that it is unrepresentable in terms of a single Ornstein-Uhlenbeck process.) Regardless, it is unimportant to us if V is a strong solution or not.⁸

6.2.3. Extended Price Formulation in Case $n = 2$. Recall W^1, W^2 are independent standard Brownian motions, set

$$\sigma(y, z, s) = \begin{bmatrix} \frac{\kappa}{2} & 0 \\ 0 & \frac{\kappa}{2} \\ \rho sy & \rho sz \end{bmatrix} \quad (6.9)$$

and define a new sde of the form:

$$d \begin{bmatrix} Y_t \\ Z_t \\ S_t^c \end{bmatrix} = \begin{bmatrix} -\frac{\rho}{2} Y_t \\ -\frac{\rho}{2} Z_t \\ \mu S_t^c \end{bmatrix} dt + \sigma(Y_t, Z_t, S_t^c) \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix}. \quad (6.10)$$

This equation has a unique strong solution. Indeed, the first two rows immediately give strong uniqueness for Y, Z and then S^c is uniquely solved as a stochastic exponential (see e.g. Protter (2004)). This solution can be rewritten as:

$$d \begin{bmatrix} Y_t \\ Z_t \\ S_t^c \end{bmatrix} = \begin{bmatrix} -\frac{\rho}{2} Y_t \\ -\frac{\rho}{2} Z_t \\ \mu S_t^c \end{bmatrix} dt + \begin{bmatrix} \frac{\frac{\kappa}{2} Z_t}{\sqrt{Y_t^2 + Z_t^2}} & \frac{\frac{\kappa}{2} Y_t}{\sqrt{Y_t^2 + Z_t^2}} \\ -\frac{\frac{\kappa}{2} Y_t}{\sqrt{Y_t^2 + Z_t^2}} & \frac{\frac{\kappa}{2} Z_t}{\sqrt{Y_t^2 + Z_t^2}} \\ 0 & \rho S_t^c V_t^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} dW_t \\ d\widehat{\beta}_t \end{bmatrix}, \quad (6.11)$$

⁸There is a famous example of H. Tanaka of a simple SDE with weak but not strong solutions.

where

$$\begin{bmatrix} dW_t \\ d\hat{\beta}_t \end{bmatrix} = \begin{bmatrix} \frac{Z_t}{\sqrt{Y_t^2 + Z_t^2}} & \frac{-Y_t}{\sqrt{Y_t^2 + Z_t^2}} \\ \frac{Y_t}{\sqrt{Y_t^2 + Z_t^2}} & \frac{Z_t}{\sqrt{Y_t^2 + Z_t^2}} \end{bmatrix} \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix}. \quad (6.12)$$

Now, the last row of (6.11) together with (6.4, 6.5, 6.6, 6.7, 6.8) show that $(S = S^i S^c, V = Y^2 + Z^2)$ is the Heston model with $\nu = \frac{\kappa^2}{2}$. Moreover, (6.9) does satisfy (2.2) since

$$(\nabla \sigma_1) \sigma_2 = \begin{pmatrix} 0 \\ 0 \\ \rho^2 s y z \end{pmatrix} = (\nabla \sigma_2) \sigma_1 \quad (6.13)$$

so we will be able to look for simple explicit solutions. Our extended Heston system (6.10) can also be written as a Stratonovich equation:

$$d \begin{bmatrix} Y_t \\ Z_t \\ S_t^c \end{bmatrix} = \begin{bmatrix} -\frac{\rho}{2} Y_t \\ -\frac{\rho}{2} Z_t \\ \mu S_t^c - \frac{\kappa \rho S_t^c}{2} - S_t^c \rho^2 \frac{Y_t^2 + Z_t^2}{2} \end{bmatrix} dt + \begin{bmatrix} \frac{\kappa}{2} & 0 \\ 0 & \frac{\kappa}{2} \\ \rho S_t^c Y_t & \rho S_t^c Z_t \end{bmatrix} \bullet \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix}, \quad (6.14)$$

where the stochastic integral implied by the \bullet is now interpreted in the Fisk-Stratonovich sense. We define the full Fisk-Stratonovich drift coefficient to be:

$$h(y, z, s, v) = \begin{bmatrix} -\frac{\rho}{2} y \\ -\frac{\rho}{2} z \\ \mu s - \frac{\kappa \rho s}{2} - s \rho^2 \frac{y^2 + z^2}{2} \end{bmatrix}. \quad (6.15)$$

Remark 11. *Reformulating the Heston equations into a higher dimensional equation so that commutator conditions like (6.13) are true and explicit solutions exist is one of our main contributions. It is believed that similar techniques can be used on some other interesting financial models.*

6.2.4. Explicit Solutions for Extended Heston in case $n = 2$. We can solve for the possible strong solutions to (6.11). The first step is to transform the equation to a simpler one using Theorem 2 of Kouritzin and Remillard (2016), restated here in the case $p = 3$ and $d = r = 2$ for convenience:

Theorem 3. *Let $D \subset \mathbb{R}^3$ be a bounded convex domain, X_0 be a random variable living in D , W be an \mathbb{R}^2 -valued standard Brownian motion and $h : D \rightarrow \mathbb{R}^3$, $\sigma : D \rightarrow \mathbb{R}^{3 \times 2}$ be twice continuously differentiable functions with $\sigma(X_0)$ having full rank and satisfying (2.2). Then, the Stratonovich SDE $dX_t = h(X_t)dt + \sigma(X_t) \bullet dW_t$ has a solution $X_t = \Lambda^{-1} \begin{pmatrix} \bar{X}_t \\ \hat{X}_t \end{pmatrix}$ on $[0, \tau]$ for some stopping time $\tau > 0$, in terms of a simpler SDE*

$$\begin{bmatrix} \bar{X}_t \\ \hat{X}_t \end{bmatrix} = \int_0^t \hat{h} \begin{pmatrix} \bar{X}_s \\ \hat{X}_s \end{pmatrix} ds + \begin{pmatrix} W_t \\ 0 \end{pmatrix} + \Lambda(X_0), \quad \text{with } \hat{h}(x) = (\nabla \Lambda h) \circ \Lambda^{-1}(x),$$

and a local diffeomorphism Λ if and only if the simpler SDE has a solution up to a stopping time at least as large as τ . Without loss of generality, the local diffeomorphism can have the form $\Lambda = \Lambda_2 \circ \Lambda_1$ for any local diffeomorphisms

$\Lambda_1 : D \rightarrow \mathbb{R}^3$ satisfying $\nabla \Lambda_1 \sigma_1 \circ \Lambda_1^{-1}(x) = e_1$ and $\Lambda_2 : \Lambda_1(D) \rightarrow \mathbb{R}^3$ satisfying $\{\nabla \Lambda_2 \nabla \Lambda_1 \sigma_2\} \circ (\Lambda_1^{-1} \circ \Lambda_2^{-1}(x)) = e_2$, where $(e_1 e_2 e_3) = I_3$ is the identity matrix.

There are three things to note:

- (1) The diffusion coefficient is just $\begin{pmatrix} I_2 \\ 0 \end{pmatrix}$ for the simpler SDE. In this case, there is no difference between the Itô and Stratonovich equations so we have just stated the simpler SDE as the more common Itô equation.
- (2) We can check this local solution to see if it is actually a global solution. We will do this below and determine that it is a global solution in our case.
- (3) We can check \hat{h} to see if these equations are solvable. We will do this below and actually solve the simplified SDE and the diffeomorphism in the extended Heston case.
- (4) It is shown in Kouritzin and Remillard (2016) that (2.2) is also necessary if we want to have such local solutions for all initial random variables X_0 .

In our Heston case $\overline{X} = (\overline{Y}, \overline{Z})'$ and $\hat{X} = \hat{S}^c$ and we can use Theorem 3 to obtain:

Theorem 4. Suppose $(W^1, W^2)'$ is a standard \mathbb{R}^2 -valued Brownian motion and $(\overline{Y}_t, \overline{Z}_t, \hat{S}_t^c)'$ is the strong solution to:

$$\begin{aligned} d \begin{bmatrix} \overline{Y}_t \\ \overline{Z}_t \end{bmatrix} &= \begin{bmatrix} -\frac{\rho}{2} \overline{Y}_t \\ -\frac{\rho}{2} \overline{Z}_t \end{bmatrix} dt + d \begin{bmatrix} W_t^1 \\ W_t^2 \end{bmatrix}, \\ d\hat{S}_t^c &= \hat{S}_t^c \left[\mu - \frac{\kappa \rho}{2} + \left[\frac{\kappa \rho \rho}{4} - \frac{\kappa^2 \rho^2}{8} \right] \{ \overline{Y}_t^2 + \overline{Z}_t^2 \} \right] dt. \end{aligned}$$

Then, $\begin{bmatrix} Y_t \\ Z_t \\ S_t^c \end{bmatrix} = \Lambda^{-1} \begin{pmatrix} \overline{Y}_t \\ \overline{Z}_t \\ \hat{S}_t^c \end{pmatrix}$ with $\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}$ satisfies (6.11, 6.12), where

$$\Lambda(x) = \begin{bmatrix} \frac{2}{\kappa} x_1 \\ \frac{2}{\kappa} x_2 \\ x_3 \exp \left(-\frac{\rho}{\kappa} (x_1^2 + x_2^2) \right) \end{bmatrix}, \quad \Lambda^{-1}(x) = \begin{bmatrix} \frac{\kappa}{2} x_1 \\ \frac{\kappa}{2} x_2 \\ x_3 \exp \left(\rho \frac{\kappa}{4} (x_1^2 + x_2^2) \right) \end{bmatrix}, \quad (6.16)$$

is a \mathcal{C}^2 -diffeomorphism on $\mathbb{R} \times \mathbb{R} \times (0, \infty)$.

Remark 12. We do not need Condition (C) for this theorem nor even for the solution of price S in terms of V below. We only need this condition to express the volatility in terms of the sums of squares of independent Ornstein-Uhlenbeck processes.

Remark 13. We only really care that we have a solution for the last rows of (6.11, 6.12) but we have to solve for all rows and then later throw away the unnecessary ones.

Remark 14. \overline{Y} and \overline{Z} are independent Ornstein-Uhlenbeck processes while \hat{S}^c just solves a linear ordinary differential equation (with coefficients depending upon the

random processes $\overline{Y}, \overline{Z}$). Hence, simulation and calculation is made easy by the explicit form of the diffeomorphism and its inverse. Notice that \widehat{S}^c has finite variation while S^c does not. The explanation for this is that the diffeomorphism Λ^{-1} brings \overline{Y} and \overline{Z} into the solution for S^c and thereby handles the quadratic variation.

Proof. The idea is to find the diffeomorphisms Λ_1, Λ_2 in Theorem 3. Solving $\frac{d}{dt}\theta(t; x) = \sigma_1(\theta(t; x))$ with σ as in (6.9) leads to

$$\frac{d}{dt}\theta(t; x) = \begin{bmatrix} \frac{\kappa}{2} \\ 0 \\ \rho \theta_1(t; x) \theta_3(t; x) \end{bmatrix} \text{ subject to } \theta(0; x) = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}, \quad (6.17)$$

and we find that $\theta_1(t; x) = \frac{\kappa}{2}t$; $\theta_2(t; x) = x_2$; $\theta_3(t; x) = x_3 \exp\left(\frac{\rho\kappa}{4}t^2\right)$. Substituting $t = x_1$ in, we have that

$$\Lambda_1^{-1}(x) = \begin{bmatrix} \frac{\kappa}{2}x_1 \\ x_2 \\ x_3 \exp\left(\frac{\rho\kappa}{4}x_1^2\right) \end{bmatrix}, \quad (6.18)$$

which has inverse

$$\Lambda_1(y) = \begin{bmatrix} \frac{2}{\kappa}y_1 \\ y_2 \\ y_3 \exp\left(-\frac{\rho}{\kappa}y_1^2\right) \end{bmatrix}. \quad (6.19)$$

Next, it follows that

$$\nabla\Lambda_1(y) = \begin{bmatrix} \frac{2}{\kappa} & 0 & 0 \\ 0 & 1 & 0 \\ -2\frac{\rho}{\kappa}y_1y_3 \exp\left(-\frac{\rho}{\kappa}y_1^2\right) & 0 & \exp\left(-\frac{\rho}{\kappa}y_1^2\right) \end{bmatrix} \quad (6.20)$$

so $\widehat{\sigma}_1(x) = \{\nabla\Lambda_1\sigma_1\}(\Lambda_1^{-1}x) = e_1$ and we have found our first diffeomorphism in Theorem 3. To find the second diffeomorphism, we set

$$\alpha_2(x) = \{\nabla\Lambda_1\sigma_2\}(\Lambda_1^{-1}x) = \begin{bmatrix} 0 \\ \frac{\kappa}{2} \\ \rho x_2x_3 \end{bmatrix}. \quad (6.21)$$

Then, solving $\frac{d}{dt}\theta(t; x) = \alpha_2(\theta(t; x))$ leads to

$$\frac{d}{dt}\theta(t; x) = \begin{bmatrix} 0 \\ \frac{\kappa}{2} \\ \rho \theta_2(t; x) \theta_3(t; x) \end{bmatrix} \text{ s.t. } \theta(0; x) = \begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix}, \quad (6.22)$$

and we find that $\theta_1(t; x) = x_1$; $\theta_2(t; x) = \frac{\kappa}{2}t$; $\theta_3(t; x) = x_3 \exp\left(\frac{\rho\kappa}{4}t^2\right)$. Substituting $t = x_2$ in and taking the inverse, we have that

$$\Lambda_2^{-1}(x) = \begin{bmatrix} x_1 \\ \frac{\kappa}{2}x_2 \\ x_3 \exp\left(\frac{\rho\kappa}{4}x_2^2\right) \end{bmatrix}, \quad \Lambda_2(y) = \begin{bmatrix} y_1 \\ \frac{2}{\kappa}y_2 \\ y_3 \exp\left(-\frac{\rho}{\kappa}y_2^2\right) \end{bmatrix}. \quad (6.23)$$

Next, it follows that

$$\nabla \Lambda_2(y) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\kappa} & 0 \\ 0 & -2\frac{\rho}{\kappa} y_2 y_3 \exp\left(-\frac{\rho}{\kappa} y_2^2\right) & \exp\left(-\frac{\rho}{\kappa} y_2^2\right) \end{bmatrix} \quad (6.24)$$

so $\widehat{\sigma}_2(x) = \{\nabla \Lambda_2 \alpha_2\}(\Lambda_2^{-1}x) = e_2$ and we indeed have our second homeomorphism in Theorem 3. Now, we find $\Lambda = \Lambda_2 \circ \Lambda_1$ gives the diffeomorphism in (6.16) and

$$\nabla \Lambda(y) = \begin{bmatrix} \frac{2}{\kappa} & 0 & 0 \\ 0 & \frac{2}{\kappa} & 0 \\ \frac{-2\frac{\rho}{\kappa} y_1 y_3}{\exp\left(\frac{\rho}{\kappa}(y_1^2 + y_2^2)\right)} & \frac{-2\frac{\rho}{\kappa} y_2 y_3}{\exp\left(\frac{\rho}{\kappa}(y_1^2 + y_2^2)\right)} & \frac{1}{\exp\left(\frac{\rho}{\kappa}(y_1^2 + y_2^2)\right)} \end{bmatrix} \quad (6.25)$$

so $\widehat{h}(x) \doteq (\nabla \Lambda)h \circ \Lambda^{-1}(x)$ in Theorem 3 satisfies

$$\widehat{h}(x) = \begin{bmatrix} -\frac{\rho}{2}x_1 \\ -\frac{\rho}{2}x_2 \\ x_3 \left[\mu - \frac{\kappa\rho}{2} + \left[\frac{\kappa\rho\rho}{4} - \frac{\kappa^2\rho^2}{8} \right] \{x_1^2 + x_2^2\} \right] \end{bmatrix}. \quad (6.26)$$

□

6.2.5. *Finishing Proof of Theorem 1 by Solving Equations in case $n = 2$.* The solution for $(\overline{Y}_t, \overline{Z}_t, \widehat{S}_t^c)'$ in Theorem 4 is: $\overline{Y}_t = \int_0^t e^{-\frac{\rho}{2}(t-u)} dW_u^1 + e^{-\frac{\rho}{2}t} \overline{Y}_0$, $\overline{Z}_t = \int_0^t e^{-\frac{\rho}{2}(t-u)} dW_u^2 + e^{-\frac{\rho}{2}t} \overline{Z}_0$ (with $\overline{Y}_0^2 + \overline{Z}_0^2 = \frac{\kappa^2}{4} V_0$ to be consistent with (6.6,6.7)), and

$$\widehat{S}_t^c = \widehat{S}_0^c \exp \left(\left[\mu - \frac{\kappa\rho}{2} \right] t + \left[\frac{\kappa\rho\rho}{4} - \frac{\kappa^2\rho^2}{8} \right] \int_0^t \{ \overline{Y}_s^2 + \overline{Z}_s^2 \} ds \right). \quad (6.27)$$

Moreover, it follows by (6.16) and (6.7) that

$$S_t^c = \widehat{S}_t^c \exp \left(\frac{\rho\kappa}{4} (\overline{Y}_t^2 + \overline{Z}_t^2) \right) = \widehat{S}_t^c \exp \left(\frac{\rho}{\kappa} (Y_t^2 + Z_t^2) \right) = \widehat{S}_t^c \exp \left(\frac{\rho}{\kappa} V_t \right)$$

and it follows by (6.27), Theorem 4, (6.16) and substitution that

$$\begin{aligned} S_t^c &= S_0^c \exp \left(\left[\mu - \frac{\kappa\rho}{2} \right] t + \left[\frac{\kappa\rho\rho}{4} - \frac{\kappa^2\rho^2}{8} \right] \int_0^t \{ \overline{Y}_s^2 + \overline{Z}_s^2 \} ds + \frac{\rho}{\kappa} (V_t - V_0) \right) \\ &= S_0^c \exp \left(\left[\mu - \frac{\kappa\rho}{2} \right] t + \left[\frac{\rho\rho}{\kappa} - \frac{\rho^2}{2} \right] \int_0^t V_s ds + \frac{\rho}{\kappa} (V_t - V_0) \right). \end{aligned} \quad (6.28)$$

We also get a solution for the simplified Heston (2.7) by computing

$$S_t^i = \exp \left(\sqrt{1 - \rho^2} \int_0^t V_s^{\frac{1}{2}} dB_s - \frac{1 - \rho^2}{2} \int_0^t V_s ds \right) \quad (6.29)$$

and then multiplying $S_t = S_t^c S_t^i$ to get (2.3) of Theorem 1 in the case $n = 2$. □

6.2.6. *Case $n \neq 2$.* Insomuch as the guess and check proof of Theorem 1 is as simple as Itô's formula, our real goal here is to motivate how this solution was actually arrived at and how weak solutions for other models might be found. With this easy Ito lemma test, a formal proof along these lines is less important. Hence, we have given all the steps just in the case $n = 2$ and we will just explain the differences required for the case $n \neq 2$ instead of going through the formal proof with these methods.

The price splitting was already done in general. There is no change there.

For the volatility in the case $n \in \{1, 3, 4, \dots\}$, we start with n independent standard Brownian motions W^1, \dots, W^n and follow Subsection 6.2.2. The differences are: We replace Y, Z with $\{Y_t^i = \frac{\kappa}{2} \int_0^t e^{-\frac{\kappa}{2}(t-u)} dW_u^i + e^{-\frac{\kappa}{2}t} Y_0^i\}_{i=1}^n$ and set

$$\hat{\beta}_t = \sum_{i=1}^n \int_0^t \frac{Y_u^i}{\sqrt{\sum_{j=1}^n (Y_u^j)^2}} dW_u^i \quad (6.30)$$

to find that $V = \sum_{i=1}^n (Y^i)^2$ satisfies (6.6) when $\nu = \frac{n\kappa^2}{4}$ (and $V_0 = \sum_{i=1}^n (Y_0^i)^2$).

For the extended price formulation when $n \in \{1, 3, 4, \dots\}$, we set

$$\sigma(y_1, \dots, y_n, s) = \begin{bmatrix} \frac{\kappa}{2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\kappa}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\kappa}{2} & 0 \\ 0 & 0 & \cdots & 0 & \frac{\kappa}{2} \\ s\rho y_1 & s\rho y_2 & \cdots & s\rho y_{n-1} & s\rho y_n \end{bmatrix} \quad (6.31)$$

and find $\nabla \sigma_i \sigma_j = (0, \dots, 0, s\rho^2 y_i y_j)'$ for $i \neq j$ so (2.2) clearly holds. (For clarity, $\sigma = (\frac{\kappa}{2}, s\rho y_1)'$ when $n = 1$.) Now, define a new sde of the form:

$$d \begin{bmatrix} Y_t^1 \\ \vdots \\ Y_t^n \\ S_t^c \end{bmatrix} = \begin{bmatrix} -\frac{\kappa}{2} Y_t^1 \\ \vdots \\ -\frac{\kappa}{2} Y_t^n \\ \mu S_t^c \end{bmatrix} dt + \sigma(Y_t^1, \dots, Y_t^n, S_t^c) \begin{bmatrix} dW_t^1 \\ \vdots \\ dW_t^n \end{bmatrix}. \quad (6.32)$$

This equation has a unique strong solution and it can be rewritten by postmultiplying σ by OO^{-1} , where

$$O = \begin{bmatrix} \frac{Y_t^n}{\sqrt{V_t}} & 0 & \cdots & 0 & \frac{Y_t^1}{\sqrt{V_t}} \\ 0 & \frac{Y_t^n}{\sqrt{V_t}} & \cdots & 0 & \frac{Y_t^2}{\sqrt{V_t}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{Y_t^n}{\sqrt{V_t}} & \frac{Y_t^{n-1}}{\sqrt{V_t}} \\ -\frac{Y_t^1}{\sqrt{V_t}} & -\frac{Y_t^2}{\sqrt{V_t}} & \cdots & -\frac{Y_t^{n-1}}{\sqrt{V_t}} & \frac{Y_t^n}{\sqrt{V_t}} \end{bmatrix}, \quad (6.33)$$

and (abusing notation by letting $Y_i = Y_t^i$)

$$O^{-1} = \begin{bmatrix} \frac{Y_2^2 + \dots + Y_n^2}{Y_n \sqrt{V_t}} & -\frac{Y_1 Y_2}{Y_n \sqrt{V_t}} & -\frac{Y_1 Y_3}{Y_n \sqrt{V_t}} & \dots & -\frac{Y_1 Y_{n-1}}{Y_n \sqrt{V_t}} & -\frac{Y_1}{\sqrt{V_t}} \\ -\frac{Y_1 Y_2}{Y_n \sqrt{V_t}} & \frac{Y_1^2 + Y_3^2 + \dots + Y_n^2}{Y_n \sqrt{V_t}} & -\frac{Y_2 Y_3}{Y_n \sqrt{V_t}} & \dots & -\frac{Y_2 Y_{n-1}}{Y_n \sqrt{V_t}} & -\frac{Y_2}{\sqrt{V_t}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{Y_1 Y_{n-1}}{Y_n \sqrt{V_t}} & -\frac{Y_2 Y_{n-1}}{Y_n \sqrt{V_t}} & -\frac{Y_3 Y_{n-1}}{Y_n \sqrt{V_t}} & \dots & \frac{Y_1^2 + \dots + Y_{n-2}^2 + Y_n^2}{Y_n \sqrt{V_t}} & -\frac{Y_{n-1}}{\sqrt{V_t}} \\ \frac{Y_1}{\sqrt{V_t}} & \frac{Y_2}{\sqrt{V_t}} & \frac{Y_3}{\sqrt{V_t}} & \dots & \frac{Y_{n-1}}{\sqrt{V_t}} & \frac{Y_n}{\sqrt{V_t}} \end{bmatrix}, \quad (6.34)$$

as:

$$d \begin{bmatrix} Y_t^1 \\ \vdots \\ Y_t^n \\ S_t^c \end{bmatrix} = \begin{bmatrix} -\frac{\varrho Y_t^1}{2} \\ \vdots \\ -\frac{\varrho Y_t^n}{2} \\ \mu S_t^c \end{bmatrix} dt + \begin{bmatrix} \frac{\kappa}{2} \frac{Y_t^n}{\sqrt{V_t}} & 0 & \dots & 0 & \frac{\kappa}{2} \frac{Y_t^1}{\sqrt{V_t}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{\kappa}{2} \frac{Y_t^n}{\sqrt{V_t}} & \frac{\kappa}{2} \frac{Y_t^{n-1}}{\sqrt{V_t}} \\ -\frac{\kappa}{2} \frac{Y_t^1}{\sqrt{V_t}} & -\frac{\kappa}{2} \frac{Y_t^2}{\sqrt{V_t}} & \dots & -\frac{\kappa}{2} \frac{Y_t^{n-1}}{\sqrt{V_t}} & \frac{\kappa}{2} \frac{Y_t^n}{\sqrt{V_t}} \\ 0 & 0 & \dots & 0 & \rho S_t^c V_t^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} dA_t^1 \\ \vdots \\ dA_t^{n-1} \\ d\hat{\beta}_t \end{bmatrix}, \quad (6.35)$$

where $(A^1, \dots, A^{n-1}, \hat{\beta})' = O^{-1}(W^1, \dots, W^n)'$ so $\hat{\beta}$ does satisfy (6.30). This extended Heston solution (6.32) can also be written in Fisk-Stratonovich form as

$$d \begin{bmatrix} Y_t^1 \\ \vdots \\ Y_t^n \\ S_t^c \end{bmatrix} = \begin{bmatrix} -\frac{\varrho}{2} Y_t^1 \\ \vdots \\ -\frac{\varrho}{2} Y_t^n \\ \left(\mu - \frac{n\kappa\rho}{4}\right) S_t^c - S_t^c \rho^2 \frac{(Y_t^1)^2 + \dots + (Y_t^n)^2}{2} \end{bmatrix} dt + \sigma(Y_t^1, \dots, Y_t^n, S_t^c) \bullet \begin{bmatrix} dW_t^1 \\ \vdots \\ dW_t^n \end{bmatrix}, \quad (6.36)$$

from which we can apply Theorem 2 of Kouritzin and Remillard (2016) (knowing (2.2) holds) in the case $p = n + 1$ and $d = r = n$ to find (6.36) has a strong solution up to some stopping time $\tau > 0$ if and only if

$$d \begin{bmatrix} \bar{Y}_t^1 \\ \vdots \\ \bar{Y}_t^n \end{bmatrix} = \begin{bmatrix} -\frac{\varrho}{2} \bar{Y}_t^1 \\ \vdots \\ -\frac{\varrho}{2} \bar{Y}_t^n \end{bmatrix} dt + d \begin{bmatrix} W_t^1 \\ \vdots \\ W_t^n \end{bmatrix}, \quad (6.37)$$

$$d\hat{S}_t^c = \hat{S}_t^c \left[\mu - \frac{n\kappa\rho}{4} + \left[\frac{\kappa\rho\varrho}{4} - \frac{\kappa^2\rho^2}{8} \right] \left\{ (\bar{Y}_t^1)^2 + \dots + (\bar{Y}_t^n)^2 \right\} \right] dt \quad (6.38)$$

does. Moreover, the solutions to (6.36) and (6.37,6.38) satisfy $\begin{bmatrix} Y_t^1 \\ \vdots \\ Y_t^n \\ S_t^c \end{bmatrix} = \Lambda^{-1} \begin{pmatrix} \bar{Y}_t^1 \\ \vdots \\ \bar{Y}_t^n \\ \widehat{S}_t^c \end{pmatrix}$,

where \mathcal{C}^2 -diffeomorphism Λ is given by

$$\Lambda(x) = \begin{bmatrix} \frac{2}{\kappa} x_1 \\ \vdots \\ \frac{2}{\kappa} x_n \\ x_{n+1} \exp \left(-\frac{\rho}{\kappa} (x_1^2 + \cdots + x_n^2) \right) \end{bmatrix}, \quad \Lambda^{-1}(x) = \begin{bmatrix} \frac{\kappa}{2} x_1 \\ \vdots \\ \frac{\kappa}{2} x_n \\ x_{n+1} \exp \left(\rho \frac{\kappa}{4} (x_1^2 + \cdots + x_n^2) \right) \end{bmatrix}. \quad (6.39)$$

The solution to (6.37,6.38) is then

$$\bar{Y}_t^i = \int_0^t e^{-\frac{\rho}{2}(t-u)} dW_u^i + e^{-\frac{\rho}{2}t} \bar{Y}_0^i, \quad i = 1, \dots, n \text{ and} \quad (6.40)$$

$$\widehat{S}_t^c = \widehat{S}_0^c \exp \left(\left[\mu - \frac{n\kappa\rho}{4} \right] t + \left[\frac{\kappa\rho\rho}{4} - \frac{\kappa^2\rho^2}{8} \right] \int_0^t \left\{ (\bar{Y}_s^1)^2 + \cdots + (\bar{Y}_s^n)^2 \right\} ds \right) \quad (6.41)$$

from which it follows using (6.39) that

$$S_t^c = S_0^c \exp \left(\left[\mu - \frac{n\kappa\rho}{4} \right] t + \left[\frac{\rho\rho}{\kappa} - \frac{\rho^2}{2} \right] \int_0^t V_s ds + \frac{\rho}{\kappa} (V_t - V_0) \right) \quad (6.42)$$

with $V_t = \frac{\kappa^2}{4} \left\{ (\bar{Y}_t^1)^2 + \cdots + (\bar{Y}_t^n)^2 \right\}$. The result follows by multiplying $S_t = S_t^i S_t^c$ and Itô's formula. \square

6.3. Proof of Theorem 2. By Theorem 1, $(\widehat{S}, \widehat{V})$, defined in (2.9,2.10) satisfies the Heston model with parameters ν_κ, μ_κ defined in (2.8). Hence, by (6.3)

$$\begin{aligned} M_t(f) &= f(\widehat{S}_t, \widehat{V}_t) - \int_0^t \mu_\kappa \widehat{S}_u \partial_s f(\widehat{S}_u, \widehat{V}_u) + (\nu_\kappa - \rho \widehat{V}_u) \partial_v f(\widehat{S}_u, \widehat{V}_u) \\ &\quad + \frac{1}{2} \widehat{S}_u^2 \widehat{V}_u \partial_s^2 f(\widehat{S}_u, \widehat{V}_u) + \rho \kappa \widehat{S}_u \widehat{V}_u \partial_s \partial_v f(\widehat{S}_u, \widehat{V}_u) + \frac{1}{2} \kappa^2 \widehat{V}_u \partial_v^2 f(\widehat{S}_u, \widehat{V}_u) du \end{aligned} \quad (6.43)$$

(for $f \in \mathcal{S}(\mathbb{R}^2)$, the rapidly decreasing functions) has the following P -martingale representation

$$\begin{aligned} M_t(f) &= \int_0^t [\kappa \partial_v f(\widehat{S}_u, \widehat{V}_u) + \rho \widehat{S}_u \partial_s f(\widehat{S}_u, \widehat{V}_u)] \widehat{V}_u^{\frac{1}{2}} d\widehat{\beta}_u \\ &\quad + \int_0^t \sqrt{1 - \rho^2} \widehat{S}_u \partial_s f(\widehat{S}_u, \widehat{V}_u) \widehat{V}_u^{\frac{1}{2}} dB_u \quad \text{with} \quad \widehat{\beta}_t = \sum_{i=1}^n \int_0^t \frac{Y_u^i}{\sqrt{\sum_{j=1}^n (Y_u^j)^2}} dW_u^i. \end{aligned} \quad (6.44)$$

Separately, it follows by Itô's formula and (2.7) that

$$\ln(\widehat{V}_t) - \ln(\widehat{V}_0) = \int_0^t \frac{\nu_\kappa - \rho \widehat{V}_s}{\widehat{V}_s} ds + \int_0^t \frac{\kappa}{\widehat{V}_s^{\frac{1}{2}}} d\widehat{\beta}_s - \frac{1}{2} \int_0^t \frac{\kappa^2}{\widehat{V}_s} ds \quad (6.45)$$

so, using (2.8), (2.11) is equivalent to

$$\widehat{L}_t = \exp \left\{ \int_0^t \frac{\nu - \nu_\kappa}{\kappa \widehat{V}_s^{\frac{1}{2}}} d\widehat{\beta}_s - \frac{1}{2} \int_0^t \frac{|\nu - \nu_\kappa|^2}{\kappa^2 \widehat{V}_s} ds \right\}. \quad (6.46)$$

It follows from (6.46) and the Novikov condition that $t \rightarrow \widehat{L}_t^{\eta_\varepsilon} \doteq \widehat{L}_{\eta_\varepsilon \wedge t}$ is an L^r -martingale for any $r > 0$. This fact will be used in the development below and to conclude $m_t(f)$ is a martingale versus just a local martingale. Next, it follows by (6.44), Itô's formula, (2.8) and the fact $d\widehat{L}_t = \widehat{L}_t \frac{\nu - \nu_\kappa}{\kappa} \widehat{V}_t^{-\frac{1}{2}} d\widehat{\beta}_t$ (by (6.46)) that the quadratic covariance satisfies

$$\begin{aligned} [\widehat{L}^{\eta_\varepsilon}, f(\widehat{S}, \widehat{V})]_t &= \int_0^{t \wedge \eta_\varepsilon} \widehat{L}_u^{\eta_\varepsilon} \frac{\nu - \nu_\kappa}{\kappa} \widehat{V}_u^{-\frac{1}{2}} \left[\kappa \partial_v f(\widehat{S}_u, \widehat{V}_u) + \rho \widehat{S}_u \partial_s f(\widehat{S}_u, \widehat{V}_u) \right] \widehat{V}_u^{\frac{1}{2}} du \\ &= \int_0^{t \wedge \eta_\varepsilon} \widehat{L}_u^{\eta_\varepsilon} \left[(\nu - \nu_\kappa) \partial_v f(\widehat{S}_u, \widehat{V}_u) + (\mu - \mu_\kappa) \widehat{S}_u \partial_s f(\widehat{S}_u, \widehat{V}_u) \right] du. \end{aligned} \quad (6.47)$$

Now, it follows by (6.43, 6.47) and integration by parts that

$$\begin{aligned} m_t(f) &= \widehat{L}_t^{\eta_\varepsilon} f(\widehat{S}_t, \widehat{V}_t) - \int_0^{t \wedge \eta_\varepsilon} \widehat{L}_u^{\eta_\varepsilon} \left[\mu \widehat{S}_u \partial_s f(\widehat{S}_u, \widehat{V}_u) + (\nu - \varrho \widehat{V}_u) \partial_v f(\widehat{S}_u, \widehat{V}_u) \right] du \\ &\quad - \int_{t \wedge \eta_\varepsilon}^t \widehat{L}_u^{\eta_\varepsilon} \left[\mu_\kappa \widehat{S}_u \partial_s f(\widehat{S}_u, \widehat{V}_u) + (\nu_\kappa - \varrho \widehat{V}_u) \partial_v f(\widehat{S}_u, \widehat{V}_u) \right] du \\ &\quad - \int_0^t \widehat{L}_u^{\eta_\varepsilon} \left[\frac{1}{2} \widehat{S}_u^2 \widehat{V}_u \partial_s^2 f(\widehat{S}_u, \widehat{V}_u) + \rho \kappa \widehat{S}_u \widehat{V}_u \partial_s \partial_v f(\widehat{S}_u, \widehat{V}_u) + \frac{1}{2} \kappa^2 \widehat{V}_u \partial_v^2 f(\widehat{S}_u, \widehat{V}_u) \right] du \end{aligned} \quad (6.48)$$

is a local martingale, which by (6.44) has form

$$\begin{aligned} m_t(f) &= \int_0^t \widehat{L}_u^{\eta_\varepsilon} \left[\kappa \partial_v f(\widehat{S}_u, \widehat{V}_u) + \rho \widehat{S}_u \partial_s f(\widehat{S}_u, \widehat{V}_u) + \frac{\nu - \nu_\kappa}{\kappa \widehat{V}_u} f(\widehat{S}_u, \widehat{V}_u) \right] \widehat{V}_u^{\frac{1}{2}} d\widehat{\beta}_u \\ &\quad + \int_0^t \widehat{L}_u^{\eta_\varepsilon} \sqrt{1 - \rho^2} \widehat{S}_u \partial_s f(\widehat{S}_u, \widehat{V}_u) \widehat{V}_u^{\frac{1}{2}} dB_u. \end{aligned} \quad (6.49)$$

(Since we have used other randomness to create the $\{Y^i\}_{i=1}^n$ we can not conclude that $m_t(f)$ is adapted to the filtration generated by β, B but it is adapted to the filtration created by B, W^1, \dots, W^n .)

Now, $\widehat{L}_t^{\eta_\varepsilon}$ and $m_t^{\eta_\varepsilon}(f) \doteq m_{t \wedge \eta_\varepsilon}(f)$ are martingales so one has by (6.48) and Fubini's theorem that

$$\begin{aligned} &\widehat{E} \left[\left(f(\widehat{S}_{t_{n+1}}, \widehat{V}_{t_{n+1}}) - f(\widehat{S}_{t_n}, \widehat{V}_{t_n}) - \int_{t_n}^{t_{n+1}} A_u f(\widehat{S}_u, \widehat{V}_u) du \right) \prod_{k=1}^n h_k(\widehat{S}_{t_k}, \widehat{V}_{t_k}) \right] \\ &= E \left[\widehat{L}_T^{\eta_\varepsilon} \left(f(\widehat{S}_{t_{n+1}}, \widehat{V}_{t_{n+1}}) - f(\widehat{S}_{t_n}, \widehat{V}_{t_n}) - \int_{t_n}^{t_{n+1}} A_u f(\widehat{S}_u, \widehat{V}_u) du \right) \prod_{k=1}^n h_k(\widehat{S}_{t_k}, \widehat{V}_{t_k}) \right] \\ &= E \left[(m_{t_{n+1}}(f) - m_{t_n}(f)) \prod_{k=1}^n h_k(\widehat{S}_{t_k}, \widehat{V}_{t_k}) \right] = 0, \end{aligned} \quad (6.50)$$

for all $0 \leq t_1 < t_2 < \dots < t_n < t_{n+1}$, $f \in \mathcal{S}(\mathbb{R}^2)$ and $h_1, \dots, h_n \in B(\mathbb{R}^2)$ (the bounded, measurables), where

$$\begin{aligned} A_u f(s, v) = & [\mu s \partial_s f(s, v) + (\nu - \varrho v) \partial_v f(s, v)] 1_{[0, \eta_\varepsilon]}(u) \\ & + [\mu_\kappa s \partial_s f(s, v) + (\nu_\kappa - \varrho v) \partial_v f(s, v)] 1_{[\eta_\varepsilon, T]}(u) \\ & + \frac{1}{2} s^2 v \partial_s^2 f(s, v) + \rho \kappa \partial_v \partial_s f(s, v) + \frac{\kappa^2}{2} \partial_v^2 f(s, v). \end{aligned} \quad (6.51)$$

Now, it follows by the argument on page 174 of Ethier and Kurtz (1986) that (S, V) satisfies the A_u -martingale problem with respect to \hat{P} \square .

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